Problem 1. Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. Suppose that there is a bijection \(f : X \to Y\) such that
\[
\frac{1}{10}d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq 10d_X(x_1, x_2)
\]
for all \(x_1, x_2 \in X\).

Show that if \(X\) is complete, then \(Y\) must also be complete.

[A function \(f : X \to Y\) is a bijection if it is one-to-one and onto. Equivalently, a function \(f : X \to Y\) is a bijection if it has an inverse \(f^{-1} : Y \to X\).

Solution: Let \(\{y_n\}\) be a Cauchy sequence in \(Y\). We have to show that \(\{y_n\}\) converges in \(Y\).

- Step 0: Define the sequence \(\{x_n\}\) in \(X\) by \(x_n = f^{-1}(y_n)\).
  Here we are using the fact that \(f\) is invertible, so \(f^{-1} : Y \to X\) is well defined.

- Step 1: Since \(\{y_n\}\) is Cauchy, \(\{x_n\}\) is Cauchy.
  Proof: Let \(\varepsilon > 0\). Since \(\{y_n\}\) is Cauchy, we can find an \(N > 0\) such that \(d_Y(y_n, y_m) < \varepsilon/10\) for all \(n, m > N\). Since \(y_n = f(x_n)\) and \(y_m = f(x_m)\), we thus see from the hypothesis that \(d_X(x_n, x_m) < \varepsilon\) for all \(n, m > N\). Thus \(x_n\) is Cauchy.

- Step 2: Since \(\{x_n\}\) is Cauchy, \(\{x_n\}\) converges.
  This is just because \(X\) is complete.

- Step 3: Since \(\{x_n\}\) converges, \(\{y_n\}\) converges.
  Proof: Let \(x_n\) converge to \(x\). Then \(d(x_n, x) \to 0\) as \(n \to \infty\). From hypothesis, we thus have \(d(f(x_n), f(x)) \to 0\) as \(n \to \infty\), so \(d(y_n, f(x)) \to 0\) as \(n \to \infty\). Thus, \(y_n\) converges.

Many of you got these steps reversed or otherwise out of order.
**Problem 2.** Let \((X, d)\) be a metric space, and let \(f : X \to \mathbb{R}\) and \(g : X \to \mathbb{R}\) be continuous functions from \(X\) to the real line \(\mathbb{R}\). Let \(\mathbb{R}^2\) be the plane with the Euclidean metric, and let \(h : X \to \mathbb{R}^2\) be the function
\[
h(x) = (f(x), g(x)).
\]
Show that \(h\) is continuous.

**Solution A (using sequential definition of continuity):** Let \(\{x_n\}\) be a sequence in \(X\) such that \(x_n \to x\) as \(n \to \infty\). We have to show that \(h(x_n) \to h(x)\) as \(n \to \infty\).

Since \(f, g\) are continuous, \(f(x_n) \to f(x)\) and \(g(x_n) \to g(x)\) as \(n \to \infty\). Thus
\[
\lim_{n \to \infty} |f(x_n) - f(x)| = \lim_{n \to \infty} |g(x_n) - g(x)| = 0.
\]

Squaring both sides and adding, then taking square roots, we get
\[
\lim_{n \to \infty} \sqrt{|f(x_n) - f(x)|^2 + |g(x_n) - g(x)|^2} = 0,
\]
so (since we are using the Euclidean metric)
\[
\lim_{n \to \infty} |h(x_n) - h(x)| = 0.
\]

Thus \(h(x_n) \to h(x)\) as desired.

**Solution B (using epsilon-delta definition of continuity):** Let \(x \in X\) and \(\varepsilon > 0\). We have to find a \(\delta > 0\) such that \(h(B(x, \delta)) \subset B(h(x), \varepsilon)\).

Since \(f\) is continuous, we can find a \(\delta_1\) such that \(f(B(x, \delta_1)) \subset B(f(x), \varepsilon/2)\). Similarly we can find a \(\delta_2\) such that \(g(B(x, \delta_2)) \subset B(g(x), \varepsilon/2)\).

Now let \(\delta\) be the minimum of \(\delta_1\) and \(\delta_2\). We claim that \(h(B(x, \delta)) \subset B(h(x), \varepsilon)\).

To see this, let \(y \in B(x, \delta)\). Then \(y \in B(x, \delta_1)\) and \(y \in B(x, \delta_2)\), so \(f(y) \in B(f(x), \varepsilon/2)\) and \(g(y) \in B(g(x), \varepsilon/2)\). So \(|f(y) - f(x)| < \varepsilon/2\) and \(|g(y) - g(x)| < \varepsilon/2\).

Since
\[
|h(y) - h(x)| = \sqrt{|f(y) - f(x)|^2 + |g(y) - g(x)|^2}
\]
we thus have
\[
|h(y) - h(x)| < \sqrt{\varepsilon^2/4 + \varepsilon^2/4} < \varepsilon
\]
so \(h(y) \in B(h(x), \varepsilon)\) as desired.

One can also use the inverse-image-of-open-sets definition of continuity, but it is somewhat cumbersome.
Problem 3.

Let $X$ be a Banach space, and let $T : X \to X$ be a bounded linear operator on $X$ such that $\|T\| < 1$. Let $x_0$ be an element of $X$. Show that there exists a unique $x \in X$ such that

$$x = x_0 + Tx.$$ 

(Hint: use the contraction principle).

Let $\Phi : X \to X$ denote the map

$$\Phi(x) = x_0 + Tx.$$ 

The problem can be rephrased as that of showing that $\Phi$ has exactly one fixed point. Since $X$ is a Banach space, it is complete, so it suffices to show that $\Phi$ is a contraction.

To verify this, we compute:

$$\|\Phi(x) - \Phi(y)\| = \|(x_0 + Tx) - (x_0 + Ty)\| = \|Tx - Ty\|$$

$$= \|T(x - y)\| \leq \|T\|\|x - y\| = c\|x - y\|$$

where $c = \|T\|$. Since $c$ doesn’t depend on $x$ or $y$ and $0 \leq c < 1$ by hypothesis, $\Phi$ is thus a contraction.

Note: It is also true that $T$ is a contraction, and many of you proved this. However, this fact is not directly helpful to the problem.
**Problem 4.** Let \((X, d)\) be a metric space, and \(E\) be a subset of \(X\). Show that the boundary \(\partial E\) of \(E\) is closed in \(X\).

[The boundary \(\partial E\) of \(E\) is defined to be the set of all points which are adherent to both \(E\) and the complement \(E^c\) of \(E\).]

Solution A: Since \(\partial E = \overline{E} \cap \overline{E^c}\), and the closure of any set is closed, \(\partial E\) is the intersection of two closed sets. Since the intersection of any collection of closed sets is closed, \(\partial E\) is therefore closed.

Solution B: Let \(x\) be adherent to \(\partial E\). Thus for every \(r > 0\), the ball \(B(x, r)\) must contain some element, say \(y\), in \(\partial E\). Now define \(s = r - d(x, y)\), so \(B(y, s)\) is contained in \(B(x, r)\). Since \(y\) is in the boundary of \(E\), it is adherent to both \(E\) and \(E^c\), so \(B(y, s)\) contains elements from both \(E\) and \(E^c\). Hence \(B(x, r)\) also contains elements from both \(E\) and \(E^c\).
Problem 5. Let \((X,d)\) be a metric space, and let \(E\) be a subset of \(X\). Show that if \(E\) is compact, then it must be closed in \(X\).

Solution A (using complete/totally bounded characterization of compactness): Since \(E\) is compact, it is complete. Now suppose that \(x \in X\) is adherent to \(E\). Then there exists a sequence \(x_n\) in \(E\) which converges to \(x\). Since convergent sequences are Cauchy, \(x_n\) must be a Cauchy sequence. Since \(E\) is complete, \(x_n\) must converge to a point in \(E\). Since a sequence cannot converge to more than one point, \(x\) must be in \(E\). Thus \(E\) contains all its adherent points and so it is closed.

Solution B (using convergent subsequence characterization of compactness): Suppose that \(x \in X\) is adherent to \(E\). Then there exists a sequence \(x_n\) in \(E\) which converges to \(x\). Since \(E\) is compact, there is a subsequence \(x_{n_1}, x_{n_2}, \ldots\) which converges in \(E\). Since \(x_n\) converges to \(x\), the subsequence must also converge to \(x\). Since a sequence cannot converge to more than one point, \(x\) must be in \(E\). Thus \(E\) contains all its adherent points.

Solution C (using open cover characterization of compactness): Suppose that \(x \in X\) is adherent to \(E\), but that \(x \notin E\). Consider the sets \(V_n = \{ y \in X : d(x,y) > 1/n \}\) for \(n = 1,2,3,\ldots\). Each of these sets is open. Since \(x \notin E\), the sets \(V_n\) cover \(E\), because every element \(y\) in \(E\) is distinct from \(x\) and so we must have \(d(x,y) > 1/n\) for at least one integer \(n\). Since \(E\) is compact, it can be covered by finitely many \(V_n\):

\[
E \subset V_{n_1} \cup V_{n_2} \cup \ldots \cup V_{n_k}.
\]

Let \(N = \max(n_1, n_2, \ldots, n_k)\). Clearly

\[
V_{n_1} \cup V_{n_2} \cup \ldots \cup V_{n_k} = \{ y \in X : d(x,y) > 1/N \}.
\]

Thus for every \(y \in E\), \(d(x,y) > 1/N\). This contradicts the assumption that \(x\) is adherent to \(E\).