Math 115AH, Spring 00, Final Exam

June 15, 2000

Name:

Student #:

Put down a nickname if you want your score posted

There are 12 problems. You have 3 hours. Do as many problems as you can in this time, skip those which you cannot solve. Each problem is worth 5 points. Some problems have sub-problems which build up on each other. You may use the result of one sub-problem for another sub-problem, even if you have not solved the first sub-problem.

Problem 1

Consider the linear map T from $\mathbb{R}^3$ to $\mathbb{R}^3$ given by the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

1. Give a basis of the kernel of T.
2. Give a basis of the range of T.
3. Give a diagonal matrix D whose entries are 0 or 1 such that there exist invertible matrices $P, Q \in M_{3\times 3}$ with $D = Q^{-1}AP$.
4. Give invertible matrices $P, Q \in M_{3\times 3}$ so that $D = Q^{-1}AP$ is satisfied for the matrix D you chose in 3.

Problem 2

Recall the addition theorem

$$\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

for all $a, b \in \mathbb{R}$.

Let V be the vector space over $\mathbb{C}$ of all functions $f : \mathbb{R} \to \mathbb{C}$ with the (usual) vector space structure given by

$$(af + g)(x) = af(x) + g(x)$$

for all $a \in \mathbb{C}$ and $f, g \in V$.

Let V' be the subset of all functions f of the form

$$f(x) = a\sin(x + b)$$

with $a \in \mathbb{C}$, $b \in \mathbb{R}$.

1. Prove that V' is a subspace of V.

2. What is the dimension of V' as vector space over $\mathbb{C}$? Prove your statement.

Problem 3

Let U, V and W be vector spaces over F where F is $\mathbb{R}$ or $\mathbb{C}$. Let $S : U \to V$ and $T : V \to W$ be two linear maps. Recall that the composition TS is defined by $(TS)(x) = T(S(x))$.

1. Prove that if TS is injective, then S is injective.
2. Prove that if TS is surjective, then T is surjective.
3. Give an example of U, V, W, T, S such that TS is bijective, S is not surjective, and T is not injective.
4. Assume that TS is bijective. Prove that S is surjective if and only if T is injective.

Problem 4

Let V be a vector space over $\mathbb{R}$ and let V' be a subspace of V. Let $n > 1$ be an integer and assume $x_1, \ldots, x_n$ are vectors in V. Assume that V' is a subset of the span of $x_1, \ldots, x_n$, and that the cosets $[x_1], \ldots, [x_n]$ span V/V'. Prove that the span of $x_1, \ldots, x_n$ is V.

Problem 5

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

1. Find a Jordan canonical form J of T.

2. Find an invertible matrix P such that $P^{-1}AP$ has the Jordan canonical form J you found in 1.

3. Which of the following three polynomials is the minimal polynomial of T?

$$(T - 1)$$

$$(T - 1)^2$$

$$(T - 1)^3$$

Problem 6

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

1. Find a Jordan canonical form J of T.
2. Find an invertible matrix P such that $P^{-1}AP$ has the Jordan canonical form J you found in 1.
3. Find the minimal polynomial of T.

Problem 7

Fix an invertible matrix $A \in M_{n\times n}(\mathbb{R})$.

Prove that for all $B \in M_{n\times n}(\mathbb{R})$ we have

$$\det(AB) = \det(A)\det(B)$$

(Hint: You may repeat the argument we gave in the course.)

Problem 8

In this problem you may use the fact that a nonzero real polynomial of degree m ,

$$\sum_{i=0}^{m} a_i x^i \quad ,$$

has at most m roots.

For $n > 1$ consider the two functions A_n and P_n mapping $\mathbb{R}^n$ to $\mathbb{R}$ defined by

$$A_n(x_1, \ldots, x_n) := \det \begin{pmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \ldots & x_2^{n-1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & x_n & x_n^2 & \ldots & x_n^{n-1} \end{pmatrix}$$

i.e., A_n is the determinant of the matrix whose ij- entry is the $(j-1)$-th power of x_i, and

$$P(x_1, \ldots, x_n) := \prod_{1 \le i,j \le n \,:\, i>j} (x_i - x_j)$$

i.e., P_n is the product of all $x_i - x_j$ with $i > j$.

1. Let $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Prove that

   $$A_n(x_1, \ldots, x_n) = 0$$

   if and only if

   $$P_n(x_1, \ldots, x_n) = 0$$

   (Hint: consider the kernel of the matrix in the definition of A_n.)

2. Now consider $x_1, \ldots, x_{n-1}$ as fixed and let $x = x_n$ vary. Prove that

   $$A_n(x_1, \ldots, x_{n-1}, x)$$

   is a polynomial in x of degree $n-1$, with highest coefficient

   $$A_{n-1}(x_1, \ldots, x_{n-1})$$

   Prove that

   $$P(x_1, \ldots, x_{n-1}, x)$$

   is a polynomial in x of degree $n-1$ with highest coefficient

   $$P_{n-1}(x_1, \ldots, x_{n-1})$$

   (Hint: For A_n use a cofactor expansion of the determinant.)

3. Prove that $P_n = A_n$ for all n.
   (Hint: Use induction and consider the roots of the polynomial $P_n(\ldots, x) - A_n(\ldots, x)$.)

Problem 9

Let $V, \langle , \rangle$ be an inner product space over $\mathbb{R}$. Let W be a subspace of V. Define $W^\perp$ to be the space of all $x \in V$ such that $\langle x, y \rangle = 0$ for all $y \in W$, i.e.,

$$W^\perp = \{x \in V : \langle x, y \rangle = 0 \text{ for all } y \in W\}$$

1. Prove that $W^\perp$ is again a subspace of V.

2. Prove that $W \cap W^\perp = \{0\}$.

3. Let $x_1, \ldots, x_k$ be an orthonormal basis of W. Define $T : V \to V$ by

   $$T(x) = \sum_{i=1}^{k} \langle x, x_i \rangle \, x_i$$

   Then T is a linear map (you may use this without proving it).

   Prove that the range of T is W and the kernel of T is $W^\perp$

4. Prove that

   $$\dim(W) + \dim(W^\perp) = \dim(V)$$

Problem 10

Let $U \in M_{n\times n}(\mathbb{C})$.

1. Write down the condition for U being unitary.
2. Assume U is unitary and λ is an eigenvalue of U. Prove that $|\lambda| = 1$.
3. Give an example of a unitary matrix in $M_{2\times 2}(\mathbb{R})$ which does not have a real eigenvalue.

Problem 11

Let $P_1(\mathbb{R})$ be the vector space of all (real) polynomials of degree less than or equal to 1. Consider the following inner product on $P_1(\mathbb{R})$:

$$\langle f, g \rangle = \int_1^3 f(x)g(x)\,dx$$

Find an orthonormal basis f_1, f_2 of $P_1(\mathbb{R})$ such that f_1 has degree 0 and f_2 has degree 1.

Problem 12

Let $A \in M_{n\times n}(\mathbb{R})$. We call A diagonalizable over $\mathbb{R}$, if there is a real invertible matrix P such that PAP^{-1} is diagonal. We call A diagonalizable over $\mathbb{C}$, if there is a complex invertible matrix P such that PAP^{-1} is diagonal.

1. Which of the following statements is true (observe that $A^T = A^*$):

   (a) If A is self adjoint, i.e., $A = A^*$, then A is diagonalizable over $\mathbb{R}$

   (b) If A is self adjoint, then A is diagonalizable over $\mathbb{C}$

   (c) If A is normal, i.e., $AA^* = A^*A$, then A is diagonalizable over $\mathbb{R}$

   (d) If A is normal, then A is diagonalizable over $\mathbb{C}$

2. Whenever the answer is no in the previous questions, give an example of a matrix with that property which is not diagonalizable over the given field.