(Partial) Solutions to Homework 1

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Though this is not a complete set of solutions, I will accept requests for the further addition of solutions.

Q1: I remark that *all* of the vector space axioms held except for (VS8). I noticed that a few people thought that since the multiplication was different this might mean that all of the scalar multiplication laws may not hold. This was not the case.

Q2: Many people made a number of rather creative examples. The following are my own favorite examples.

1. The set $\mathbb{R}u + \mathbb{R}v$, where u and v are linearly independent vectors in $\mathbb{R}^3$. Here $\mathbb{R}w = \{\alpha w | \alpha \in \mathbb{R}\}$, for a given vector w. This is closed under scalar multiplication but not vector addition.
2. The set $\mathbb{Z}^3 = \{(x_1, x_2, x_3) | x_i \in \mathbb{Z}, i = 1, 2, 3\}$, where $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ is the set of integers. This is closed under addition but not scalar multiplication.

Q3: Since we want $w \in \text{span}(\{u, v\})$ (this is true since $\text{span}(\{u, v, w\}) = \text{span}(\{u, v\})$), we can write $w = \alpha u + \beta v$ for some real numbers α, β. We want to pick u, v, w such that $\text{span}(\{u, w\}) \neq \text{span}(\{u, v, w\})$. We already have $u, w \in \text{span}(\{u, v, w\})$, so in some sense we wish to choose w so that w has nothing to do with v. With this in mind, suppose $\beta \neq 0$. Then we may divide by β, so

$$v = \left(-\frac{\alpha}{\beta}\right) u + \frac{1}{\beta} w.$$

But then $v \in \text{span}(\{u, w\})$, so $\text{span}(\{u, w\}) = \text{span}(\{u, v, w\})$. Thus we must have $w = \alpha u$ for some $\alpha \in \mathbb{R}$. Note that we cannot have $\alpha = 0$ since $\text{span}(\{v, w\}) = \text{span}(\{u, v\})$.

Finally, note that each of $\text{span}(\{u, v, w\})$, $\text{span}(\{u, v\})$, and $\text{span}(\{v, w\})$ must have dimension two: otherwise $\text{span}(\{u, w\})$ has dimension one, as does $\text{span}(\{u, v, w\})$, so $\text{span}(\{u, w\}) \subset \text{span}(\{u, v, w\})$ implies $\text{span}(\{u, w\}) = \text{span}(\{u, v, w\})$, which is contrary to what we desire to achieve. These ideas can be used to obtain a general set of vectors with the desired conditions.

More intuitively, we can appeal to our sense of dimensionality. Certainly $\text{span}(\{u, v, w\})$, $\text{span}(\{u, v\})$, and $\text{span}(\{v, w\})$ have dimension at most two. Since we want $\text{span}(\{u, w\}) \subsetneq \text{span}(\{u, v, w\})$, the dimension of $\text{span}(\{u, v, w\})$ must be two: otherwise any nontrivial subspace would be equal to the span itself. Furthermore, since $\text{span}(\{u, w\})$ is nontrivial (otherwise our vectors would not be distinct), it must have dimension n with $1 \leq n < 2$. Thus $\text{span}(\{u, w\})$ should be one dimensional. This intuition allows us to choose vectors with the desired properties.

Q7:

Definition. *If S_1 and S_2 are nonempty subsets of a vector space V, then the* **sum** *of S_1 and S_2, denoted $S_1 + S_2$, is the set $\{x + y : x \in S_1, y \in S_2\}$.*

Claim. *Let W_1 and W_2 be subspaces of a vector space V.*

1. *The set $W_1 + W_2$ is a subspace of V containing both W_1 and W_2.*
2. *Any subspace of V containing both W_1 and W_2 must also contain $W_1 + W_2$.*

Proof: 1. By Theorem 1.3 (p. 16), to see that $W_1 + W_2$ is a subspace, it suffices to show the following:

- $0 \in W_1 + W_2$
- if $x, y \in W_1 + W_2$ then $x + y \in W_1 + W_2$
- if $x \in W_1 + W_2$ and $c \in F$ then $cx \in W_1 + W_2$.

Here $0 \in W_1$ and $0 \in W_2$ so $0 + 0 = 0 \in W_1 + W_2$. If $x, y \in W_1 + W_2$, then we can write $x = x_1 + x_2$ and $y = y_1 + y_2$, where $x_1, y_1 \in W_1$ and $x_2, y_2 \in W_2$. Thus

$$\begin{aligned} x + y &= (x_1 + x_2) + (y_1 + y_2) \\ &= (x_1 + y_1) + (x_2 + y_2), \end{aligned}$$

so $x + y \in W_1 + W_2$ since $x_1 + y_1 \in W_1$ and $x_2 + y_2 \in W_2$ (because W_1, W_2 are subspaces). Similarly, if $c \in F$ and $x \in W_1 + W_2$, using our previous decomposition gives

$$\begin{aligned} cx &= c(x_1 + x_2) \\ &= (cx_1) + (cx_2), \end{aligned}$$

so $cx \in W_1 + W_2$ since $cx_1 \in W_1$ and $cx_2 \in W_2$ (again because W_1, W_2 are subspaces). It follows that $W_1 + W_2$ is a subspace of V

To see that $W_1 \subset W_1 + W_2$ (and similarly $W_2 \subset W_1 + W_2$), if $x \in W_1$ then consider $y = 0 \in W_2$. Then $x + y = x \in W_1 + W_2$, so $x \in W_1$ implies $x \in W_1 + W_2$. It follows that $W_1 \subset W_1 + W_2$.

2. Suppose that M is a subspace of V such that $W_1, W_2 \subset M$. If $x \in W_1$ and $y \in W_2$, then $x, y \in M$ and thus $x + y \in M$ since M is closed under addition. Yet then $W_1 + W_2 = \{x + y | x \in W_1, y \in W_2\} \subset M$, as claimed. □

Q8:

Claim. *If W_1 and W_2 are subspaces of a vector space V then $W_1 \cup W_2$ is a subspace iff $W_1 \subset W_2$ or $W_2 \subset W_1$.*

Proof: If $W_1 \subset W_2$, then $W_1 \cup W_2 = W_2$ is a subspace. We have a similar result for $W_2 \subset W_1$.

Suppose that $W_1 \cup W_2$ is a subspace, yet $W_1 \not\subset W_2$ and $W_2 \not\subset W_1$. Then we can find $w_1 \in W_1 \setminus W_2$ and $w_2 \in W_2 \setminus W_1$. Here $w_1, w_2 \in W_1 \cup W_2$, so $w_1 + w_2 \in W_1 \cup W_2$ since we assume that $W_1 \cup W_2$ is a subspace. Since $W_1 \cup W_2 = \{x | x \in W_1 or W_2\}$, either $w_1 + w_2 \in W_1$ or $w_1 + w_2 \in W_2$. Without loss of generality, $w_1 + w_2 \in W_1$ (the argument for $w_1 + w_2 \in W_2$ proceeds similarly). Then since $w_1, w_1 + w_2 \in W_1$, $(w_1 + w_2) - w_1 = w_2 \in W_1$ because W_1 is a subspace. Yet we chose w_2 so that $w_2 \notin W_1$, so this is a contradiction. It follows that either $W_1 \subset W_2$ or $W_1 \subset W_2$. $\square$