Solutions to the practice midterm

- Q1. To compute $[T]_\beta^\beta$, we have to compute $T(1,0)$ and $T(0,1)$. To do this we have to write $(1,0)$, $(0,1)$ as linear combinations of $(1,2)$ and $(1,1)$. A little bit of work (e.g. using Gaussian elimination) shows that

$$(1,0) = -1 \times (1,2) + 2 \times (1,1)$$

and

$$(0,1) = 1 \times (1,2) - 1 \times (1,1)$$

and hence

$$T(1,0) = -1 \times T(1,2) + 2 \times T(1,1) = -1 \times (3,2) + 2 \times (1,1)$$
$$= (-1,0) = -1 \times (1,0) + 0 \times (0,1)$$

and

$$T(0,1) = 1 \times T(1,2) - 1 \times T(1,1) = 1 \times (3,2) - 1 \times (1,1)$$
$$= (2,1) = 2 \times (1,0) + 1 \times (0,1)$$

and so

$$[T]_\beta^\beta = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}.$$

- Q2. Let v be a vector in V. Our task is to show that v is a linear combination of $v_1, \ldots, v_n$.

- Since v is in V, Tv is in W. By hypothesis, $Tv_1, \ldots, Tv_n$ span W. Thus Tv is a linear combination of $Tv_1, \ldots, Tv_n$, i.e. we can write

$$Tv = a_1 Tv_1 + a_2 Tv_2 + \ldots + a_n Tv_n$$

for some scalars $a_1, a_2, \ldots, a_n$.

- We can rewrite this as

$$Tv = T(a_1 v_1 + a_2 v_2 + \ldots + a_n v_n),$$

so that

$$T(v - a_1 v_1 - a_2 v_2 - \ldots - a_n v_n) = 0.$$

By definition of null space $N(T)$, this means that

$$v - a_1 v_1 - a_2 v_2 - \ldots - a_n v_n \in N(T).$$

But since the span of $v_1, \ldots, v_n$ contains $N(T)$, we thus see that

$$v - a_1 v_1 - a_2 v_2 - \ldots - a_n v_n$$

is in the span of $v_1, \ldots, v_n$, and must therefore be some linear combination of $v_1, \ldots, v_n$:

$$v - a_1 v_1 - a_2 v_2 - \ldots - a_n v_n = b_1 v_1 + \ldots + b_n v_n,$$

which can be rearranged as

$$v = (a_1 + b_1) v_1 + (a_2 + b_2) v_2 + \ldots + (a_n + b_n) v_n,$$

which implies that v is a linear combination of $v_1, \ldots, v_n$ as desired.

- Q3. The map $T_3 T_2$ maps V_2 to V_4. By the dimension theorem, we have

  $$\text{rank}(T_3 T_2) + \text{nullity}(T_3 T_2) = \dim(V_2) = 5,$$

  so in particular

  $$\text{rank}(T_3 T_2) \leq 5.$$

  Thus

  $$\dim(R(T_3 T_2)) \leq 5.$$

  Now we observe that $R(T_3 T_2 T_1) = \{T_3 T_2 T_1 v : v \in V_1\}$ is a subset of $R(T_3 T_2) = \{T_3 T_2 w : w \in V_2\}$. This is because every vector of the form $T_3 T_2 T_1 v$, where $v \in V_1$ is automatically also of the form $T_3 T_2 w$ for some $w \in V_2$ (just by setting $w := T v_1$) and is thus also in $R(T_3 T_2)$. Thus

  $$\dim(R(T)) = \dim(R(T_3 T_2 T_1)) \leq 5.$$

  On the other hand, if $T : V_1 \to V_4$ was surjective, then $R(T) = V_4$, and so $\dim R(T) = 6$, a contradiction. Thus T is not surjective.

- Q4. To show that $\{\sin(x), \cos(x)\}$ is a basis for W, we need to show that this set is linearly independent, and that it spans.

- To show that it is linearly dependent, suppose for contradiction that we could find a_1 and a_2, not both zero, such that the function $a_1 \sin(x) + a_2 \cos(x)$ was equal to zero. In particular, it is zero when $x = 0$, so that

$$a_1 \sin(0) + a_2 \cos(0) = 0$$

or in other words that $a_2 = 0$. It is also zero when $x = \pi/2$, so that

$$a_1 \sin(\pi/2) + a_2 \cos(\pi/2) = 0$$

or in other words that $a_1 = 0$. Thus $a_1 = a_2 = 0$, a contradiction. Thus $\{\sin(x), \cos(x)\}$ is linearly independent.

- Now we show that $\{\sin(x), \cos(x)\}$ spans W. This means we need to start with a typical element of W, say

$$a \sin(x + b) + c \cos(x + d)$$

and write it as a linear combination of $\sin(x)$ and $\cos(x)$. But by the sine and cosine addition formulae we have

$$a \sin(x+b)+c \cos(x+d) = a \sin(x) \cos(b)+a \cos(x) \sin(b)+c \cos(x) \cos(d)-c \sin(x) \sin(d)$$

which can be rearranged as

$$[a \cos(b) - c \sin(d)] \sin(x) + [a \sin(b) + c \cos(d)] \cos(x)$$

and this is clearly a linear combination of $\sin(x)$ and $\cos(x)$, as desired.

- Q5. From the dimension theorem it will suffice to compute the nullity of T, since the rank and nullity add up to 3.

- To compute the nullity, we first compute the null space. A vector $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is in the null space if

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or in other words that

$$x_1 + x_2 + 2x_3 = 0$$

$$2x_1 + x_2 + 3x_3 = 0$$

$$3x_1 + x_2 + 4x_3 = 0.$$

Now we do some Gaussian elimination. Subtracting two copies of the first row from the second, and subtracting three copies of the first row from the third, we obtain

$$x_1 + x_2 + 2x_3 = 0$$

$$-x_2 - x_3 = 0$$

$$-2x_2 - 2x_3 = 0.$$

Multiplying the second row by -1, and then adding two copies of that row to the third, we get

$$x_1 + x_2 + 2x_3 = 0$$

$$x_2 + x_3 = 0$$

$$0 = 0.$$

We can write this in terms of x_3 as

$$x_2 = -x_3; \quad x_1 = -x_3.$$

Thus the null space is given by

$$N(T) = \{ \begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \end{pmatrix} : x_3 \in \mathrm{R}\}$$

(indeed, one can easily check that every vector of the form $\begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \end{pmatrix}$ gets sent to zero by T). This space is one-dimensional (indeed, it is the span of $\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \}$), and so the nullity of T is 1. Hence the rank is 2.