Solutions to the practice final

- Q1(a). Clearly S is a subset of P_2, since all of its elements lie in P_2 by definition. To verify that it is a subspace, we have to check that S is closed under addition and scalar multiplication.

  Let us check S is closed under addition. If $p(x)$ and $q(x)$ are elements of S, then p, q lie in P_2 and $p(0) + p''(0) = q(0) + q''(0) = 0$. Thus the polynomial $p + q$ also lies in P_2, and

  $$(p+q)(0) + (p+q)''(0) = p(0) + p''(0) + q(0) + q''(0) = 0 + 0 = 0$$

  and so $p + q$ also lies in S. Thus S is closed under addition.

  Now let us check S is closed under scalar multiplication. If $p(x)$ is an element of S and c is a scalar, then p lies in P_2 and $p(0) + p''(0) = 0$. Thus the polynomial cp also lies in P_2, and

  $$(cp)(0) + (cp)''(0) = c(p(0) + p''(0)) = c0 = 0$$

  and so cp also lies in S. Thus S is closed under scalar multiplication.

  Since S lies in P_2, and is closed under both addition and scalar multiplication, it is a subspace of P_2.

- Q1(b). Before we begin, let us understand the space S better. An element p of S lies in P_2, so is of the form $p(x) = ax^2 + bx + c$ for some scalars a, b, c. However, not every such polynomial lies in S, because we have the condition $p(0) + p''(0) = 0$. Since $p'(x) = 2ax + b$ and $p''(x) = 2a$, we see that $p(0) + p''(0) = c + 2a$. Thus we must have $c + 2a = 0$ in order for p to lie in S. In summary, S consists of all the polynomials $ax^2 + bx + c$ for which $c + 2a$ is equal to 0. In other words, c is equal to $-2a$, and we can write p as the form $ax^2 + bx - 2a$.

  We can now see by inspection that p is nothing more than a linear combination of the polynomials $x^2 - 2$ and x, indeed $p = a(x^2 - 2) + bx$. Thus S is nothing more than the space of all linear combinations of $x^2 - 2$ and x. Since $x^2 - 2$ and x are clearly linearly independent (neither of them is a scalar multiple of the other), we see that S is thus two-dimensional, and has $\{x^2 - 2, x\}$ as a basis.

There are of course many other bases for S. One way to find a basis is simply to start with a single non-zero element of S, e.g. $x^2 + x - 2$. This does not span S, so we then add another element not in the span of the previous element, e.g. $x^2 + 2x - 2$. Then we can check that these two elements span (every polynomial in S can be written as a linear combination of x^2+x-2 and x^2+2x-2, and so $\{x^2+x-2, x^2+2x-2\}$ is a basis for S.

- Q2(a) We need to check whether T preserves addition and also preserves scalar multiplication.

  First let's see if T preserves addition. We need to check that $T(A+B) = TA + TB$ for all matrices A, B in $M_{2\times 2}(\mathrm{R})$. But

  $$T(A+B) = Q(A+B) = QA + QB = TA + TB$$

  so T does indeed preserve addition.

  Now let's see if T preserves scalar multiplication. We need to check that $T(cA) = cTA$ for all scalars c and all matrices A in $M_{2\times 2}(\mathrm{R})$. But

  $$T(cA) = Q(cA) = c(QA) = cTA$$

  so T does indeed preserve scalar multiplication. Thus T is a linear transformation.

- Q2(b) The kernel (or nullspace) of T is the set of all matrices A in $M_{2\times 2}(\mathrm{R})$ for which $TA = 0$; note that 0 here means the 2×2 zero matrix, since TA is a 2×2 matrix. If we write

  $$A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

  then we see that

  $$TA = QA = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2a & 2b \\ 3c & 3d \end{pmatrix}$$

  so if $TA = 0$, then $2a = 2b = 3c = 3d = 0$, which implies that $a = b = c = d = 0$, which implies that $A = 0$. Thus the only matrix A which lies in the null space of T is the zero matrix, so $Ker(T) = N(T) = \{0\}$. (Another way to see this is just to use the fact that Q is an invertible matrix (it has non-zero determinant), so that if $QA = 0$, this forces $Q^{-1}QA = A = 0$.)

- Q2(c). $M_{2\times 2}(\mathrm{R})$ is four-dimensional, and the null space is 0-dimensional, so by the dimension theorem the range $R(T)$ is four-dimensional. But $R(T)$ is a subspace of $M_{2\times 2}(\mathrm{R})$, which is also four-dimensional. Hence $R(T)$ must be all of $M_{2\times 2}(\mathrm{R})$. There are many bases for this space; the standard basis is

$$\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\}.$$

- Q2(d). Because the nullity of T is zero, T is one-to-one. Because the range $R(T)$ of T is equal to all of $M_{2\times 2}(\mathrm{R})$, T is onto. Since T is one-to-one and onto, it is an isomorphism.

- Q3. There do not appear to be any obvious trig identities showing that $\cos(x)$, $\cos(2x)$, and $\cos(3x)$ are linearly dependent, so let's try to show that they are linearly independent. Suppose for contradiction that we could find scalars a, b, c, not all zero, such that $a\cos(x) + b\cos(2x) + c\cos(3x)$ is the zero function, i.e.

$$a\cos(x) + b\cos(2x) + c\cos(3x) = 0 \text{ for all } x.$$

To use this, let's plug in a few values of x. If $x = 0$, we obtain

$$a + b + c = 0.$$

If $x = \pi$, we obtain

$$-a + b - c = 0.$$

If $x = \pi/2$, we obtain

$$b = 0.$$

If $x = \pi/3$, we obtain

$$\frac{1}{2}a - \frac{1}{2}b - c = 0.$$

Combining all these facts and using a bit of Gaussian elimination we see that $a = b = c = 0$. So the only linear combination of $\cos(x)$, $\cos(2x)$, $\cos(3x)$ which gives the zero function is the zero linear combination, and so these three functions are linearly independent.

- Q4(a). We first work out the characteristic polynomial of A (which will depend on c, of course). This is

$$f(\lambda) = \det \begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & c & -\lambda \end{pmatrix} = (1-\lambda)(\lambda^2 - c).$$

This has roots at $\lambda = 1$ and $\lambda = \pm\sqrt{c}$, so these are the eigenvalues.

- Q4(b). The answer to this question depends on whether one wants to diagonalize over the reals or the complexes. Let's first suppose we are diagonalizing over the reals.

The first thing we need is for the characteristic polynomial to split; this will only happen if c is positive or zero. So when c is negative we cannot diagonalize over the reals.

Now suppose c is zero or positive. If there no repeated roots then the matrix is diagonalizable. The roots $\lambda = 1$, $\lambda = +\sqrt{c}$, and $\lambda = -\sqrt{c}$. Thus we have diagonalizability unless $c = 0$ or $c = 1$.

Let's look at the $c = 0$ case; here the characteristic polynomial has a single eigenvalue at 1 and a repeated eigenvalue at 0. To see whether it is diagonalizable, we have to look at the eigenspaces and see if there are enough eigenvectors to span R^3. First let us look at the eigenvectors with eigenvalue 1. These eigenvectors are the solution to the equation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

A little Gaussian elimination shows that this equation is satisfied when $y = z = 0$ and x is arbitrary. So the eigenvectors here are just the elements $\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$ of the x-axis.

Now let us look at the eigenvectors with eigenvalue 0. These are the solutions to the equation

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

A little Gaussian elimination shows that this equation is satisfied when $x = z = 0$ and y is arbitrary, so the eigenvectors here are just the elements $\begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$ of the y-axis.

Thus the only eigenvectors of this matrix we have are those on the x and y axes. These are not enough to span all of R^3 (both axes are only one-dimensional, so together they can only hope to span a two-dimensional space at most), so this matrix is not diagonalizable.

Now we look at the $c = 1$ case. In this case the eigenvalues are $+1$ and -1. The eigenvectors with eigenvalue $+1$ are the solutions to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix};$$

one can check that this equation is obeyed when $y = z$ and x is arbitrary. So the eigenvectors here are a two-dimensional space, consisting of all vectors of the form $\begin{pmatrix} x \\ y \\ y \end{pmatrix}$. On the other hand, the eigenvectors with eigenvalue -1 are the solutions to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = - \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

which one can check is obeyed when $x = 0$ and $y = -z$, so this is a one-dimensional space consisting of vectors of the form $\begin{pmatrix} 0 \\ y \\ -y \end{pmatrix}$. So there seem to be enough vectors here to form a basis of R^3; for instance, we can use $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ as a basis of eigenvectors (the first two have eigenvalue 1, the third has eigenvalue -1). So A is diagonalizable when $c = 1$.

To summarize, when the field of scalars is the reals, A is diagonalizable when c is positive, but not when c is 0 or negative.

When the field of scalars, the analysis is the same except when c is negative. Now the characteristic polynomial $(1-\lambda)(\lambda^2 - c)$ does split into three distinct roots ($\lambda = 1$, and the two imaginary roots $\lambda = \pm i\sqrt{-c}$), and so we now have diagonalizability when c is negative.

- Q5(a) If λ is an eigenvalue of M, then we have an eigenvector v for which $Mv = \lambda v$. Applying M to both sides we obtain

$$M^2 v = M\lambda v = \lambda M v = \lambda\lambda v = \lambda^2 v.$$

Applying M again we obtain

$$M^3 v = M\lambda^2 v = \lambda^2 M v = \lambda^2 \lambda v = \lambda^3 v.$$

Continuing in this manner, using induction, we see that

$$M^k v = \lambda^k v$$

for $k = 1, 2, 3, \ldots$. Thus v is also an eigenvector of M^k with eigenvalue λ^k. Thus the eigenvectors of M^k here are the same as those of M. (Actually, that isn't quite true; it might be that M^k has some additional eigenvectors that don't come from this argument. The most we can say right now is that the eigenspace of M^k with eigenvalue λ^k contains, but might not be equal to, the eigenspace of M with eigenvalue λ).

- Q5(b). Suppose for contradiction that A had a non-zero eigenvalue $\lambda \neq 0$. Then by Q5(a), A^r would have an eigenvalue λ^r. But A^r is the zero matrix, and clearly the only eigenvalues of the zero matrix are 0. Thus $\lambda^r = 0$, and so $\lambda = 0$, contradiction. Thus the only possible eigenvalue of A is 0.

- Q6. Note: this question is quite tricky!

- Q6(a). First suppose that V is the direct sum of W_1 and W_2. This means that $V = W_1 + W_2$ and that $W_1 \cap W_2 = \{0\}$. We have to show that every vector v in V can be written as $v = x + y$ where $x \in W_1$ and $y \in W_2$ are unique.

Let $v \in V$. Since $V = W_1 + W_2$, we know that v can be written in the form $v = x + y$ where $x \in W_1$ and $y \in W_2$. This looks like what we want, but we also need x and y to be unique. Suppose for contradiction that there was another way to write v in this form, e.g. $v = x' + y'$ where $x' \in W_1$ and $y' \in W_2$, where x', y' were not the same as x, y. Then $x + y = x' + y'$, which means that $x - x' = y' - y$. But $x - x'$ lies in W_1 and $y' - y$ lies in W_2, so both $x - x'$ and $y' - y$ must lie in $W_1 \cap W_2 = \{0\}$. Thus $x - x' = y' - y = 0$, and so $x = x'$ and $y = y'$, contradiction. Thus there is only one way to write $v = x + y$ in this form.

Now suppose instead that every vector v in V can be written as $v = x+y$ where $x \in W_1$ and $y \in W_2$ are unique. Then we have $V = W_1 + W_2$ because every vector v in V can be written as the sum of a vector in W_1 and a vector in W_2. Now we show that $W_1 \cap W_2 = \{0\}$. Suppose for contradiction that $W_1 \cap W_2$ contained a non-zero vector v. Then v lies in both W_1 and W_2, as does 0. Thus we can write $v = x + y$, where $x \in W_1$ and $y \in W_2$ in at least two different ways: $v = 0 + v$ and $v = v + 0$. But this is a contradiction. Thus $W_1 \cap W_2 = 0$.

- Q6(b). First suppose that V is the direct sum of W_1 and W_2. Then every $v \in V$ can be written as $v = x + y$ in exactly one way, where $x \in W_1$ and $y \in W_2$. Define the transformation $T : V \to V$ by setting $Tv := y$, i.e. we set Tv to be the W_2 component of the decomposition of v. We now have to check that T obeys all the desired properties.

  First, we check that T is linear. Suppose that $v, v' \in V$; we have to check that $T(v + v') = Tv + Tv'$. Write $v = x + y$ and $v' = x' + y'$, where $x, x' \in W_1$ and $y, y' \in W_2$. Then $v + v' = (x + x') + (y + y')$. Since $x + x' \in W_1$ and $y + y' \in W_2$, we see that we have decomposed $v + v'$ into a sum of something in W_1 and something in W_2. Since there is only one way to do this, we see that $y + y'$ must match $T(v + v')$, as desired. Thus T preserves addition. A similar argument shows that T preserves scalar multiplication, so that T is linear.

  Now we show that $N(T) = W_1$. First we show that every element of $N(T)$ is also in W_1. Let $v \in N(T)$; then $Tv = 0$. Thus when we decompose $v = x + y$, we have $y = Tv = 0$, so $v = x$. But x lies in W_1, hence v lies in W_1. Conversely, we need to show that every element of

W_1 lies in $N(T)$. If $v \in W_1$, then the decomposition $v = v + 0$ splits v as the sum of a vector in W_1 and a vector in W_2. Thus by definition of T, $Tv = 0$, and so v is in $N(T)$.

Now we show that $R(T) = W_2$. First we show that every element of $R(T)$ is also in W_2. Every vector in $R(T)$ is of the form Tv for some $v \in V$, and hence is equal to the vector y in the unique decomposition $v = x + y$ of v into the sum of a vector x in W_1 and y in W_2. But since Tv is equal to y, it lies in W_2. Conversely, we need to show that every vector in W_2 lies in $R(T)$. Let v be a vector in W_2. Then the decomposition $v = 0 + v$ splits v into the sum of a vector x in W_1 and y in W_2, thus $Tv = v$ by the definition of T. Thus v lies in the range of T.

Now we show that $T^2 = T$. Let v be any vector in V; we need to show that $T^2 v = Tv$. Decompose $v = x + y$, where $x \in W_1$ and $y \in W_2$; we have by definition that $Tv = y$. Since we can decompose $y = 0 + y$ we also have that $Ty = y$. Thus $T^2 v = Ty = y = Tv$ as desired.

Now we have to prove the other direction of 6(b), i.e. we assume that there is a linear transformation T with the properties listed, and we want to prove that V is the direct sum of W_1 and W_2. By part (a), we need to show that $V = W_1 + W_2$ and that $W_1 \cap W_2 = \{0\}$.

First we show that $V = W_1 + W_2$, i.e. every vector v can be written in the form $v = x + y$ for some $x \in W_1$ and $y \in W_2$. To do this, we write $v = (v - Tv) + Tv$. Clearly Tv is in the range of T and is hence in W_2 by assumption. Also, note that $T(v - Tv) = Tv - T^2 v = Tv - Tv = 0$ by assumption, so that $v = Tv$ is in the null space of T and is hence in W_1 by assumption. Thus the decomposition $v = (v - Tv) + Tv$ splits v as the sum of a vector in W_1 and a vector in W_1, and hence $v \in W_1 + W_2$ as desired.

Now we show that $W_1 \cap W_2 = \{0\}$. Suppose for contradiction that there was a non-zero vector v in $W_1 \cap W_2$. Since v lies in W_2, it is in the range of T by assumption, thus $v = Tw$ for some vector w. On the other hand, since v lies in W_1, it is in the null space of T by assumption, thus $Tv = 0$. Substituting $v = Tw$ into this we obtain $T^2 w = 0$. But since $T^2 = T$, we thus have $Tw = 0$, so $v = 0$, a contradiction. Thus $W_1 \cap W_2$ can only contain the zero vector.