Assignment 5 Due November 7 Covers: Sections 2.4-2.5

- Q1. Do exercise 15(b) of Section 2.4 of the textbook. (Note that part (a) of this exercise was already done in Q8(b) of Assignment 3).
- Q2. Do exercise 1(abde) of Section 2.5 in the textbook.
- Q3. Do exercise 2(b) of Section 2.5 in the textbook.
- Q4. Do exercise 4 of Section 2.5 in the textbook.
- Q5. Do exercise 10 of Section 2.5 in the textbook.
- Q6. Let $\beta := ((1,0),(0,1))$ be the standard basis of $\mathbf{R}^2$, and let $\beta' := ((3,-4),(4,3))$ be another basis of $\mathbf{R}^2$. Let l be the line connecting the origin to $(4,3)$, and let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the operation of reflection through l (so if $v \in \mathbf{R}^2$, then Tv is the reflected image of v through the line l.
- (a) What is $[T]_{\beta'}^{\beta'}$? (You should do this entirely by drawing pictures).
- (b) Use the change of variables formula to determine $[T]_{\beta}^{\beta}$.
- (c) If $(x,y) \in \mathbf{R}^2$, give a formula for $T(x,y)$.
- Q7. Let $T : P_n(\mathbf{R}) \to \mathbf{R}^{n+1}$ be the map

  $$T(f) := (f(0), f(1), f(2), \ldots, f(n)).$$

  Thus, for instance if $n = 3$, then $T(x^2) = (0,1,4,9)$.
- (a) Prove that T is linear.
- (b) Prove that T is an isomorphism.
- Q8*. Let A, B be $n \times n$ matrices such that $AB = I_n$, where I_n is the $n \times n$ identity matrix.
- (a) Show that $L_A L_B = I_{\mathbf{R}^n}$, where $I_{\mathbf{R}^n}$ the identity on $\mathbf{R}^n$.
- (b) Show that L_B is one-to-one and onto. (**Hint:** Use (a) to obtain the one-to-one property. Then use the Dimension theorem to deduce the onto property).

- (c) Show that $L_B L_A = I_{\mathbf{R}^n}$. (**Hint:** First use (a) to show that $L_B L_A L_B = L_B$, and then use the fact that L_B is onto).

- (d) Show that $BA = I_n$.

- (To summarize the result of this problem: if one wants to show that two $n \times n$ matrices A, B are inverses, one only needs to show $AB = I_n$; the other condition $BA = I_n$ comes for free).

- Q9. Let A, B, C be $n \times n$ matrices.

- (a) Show that A is similar to A.

- (b) Show that if A is similar to B, then B is similar to A.

- (c) Show that if A is similar to B, and B is similar to C, then A is isomorphic to C.

- (Incidentally, the above three properties (a)-(c) together mean that similarity is an *equivalence relation*).

- Q10*. Let V be a finite-dimensional vector space, let $T : V \to V$ be a linear transformation, and let $S : V \to V$ be an invertible linear transformation.

- (a) Prove that $\mathbf{R}(STS^{-1}) = S(\mathbf{R}(T))$ and $\mathbf{N}(STS^{-1}) = S(\mathbf{N}(\mathbf{T}))$. (Recall that $\mathbf{R}(T) := \{Tv : v \in V\}$ is the range of T, while $\mathbf{N}(T) := \{v \in V : Tv = 0\}$ is the null space of T. Also, for any subspace W of V, recall that $S(W) := \{Sv : v \in W\}$ is the image of W under S.)

- (b) Prove that $\text{rank}(\mathbf{R}(T)) = \text{rank}(\mathbf{R}(STS^{-1}))$ and $\text{nullity}(\mathbf{R}(T)) = \text{nullity}(\mathbf{R}(STS^{-1}))$. (**Hint:** use part (a) as well as Q1).