(Partial) Solutions to Homework 8

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Q4:

Claim (Parseval's Identity). *If $\{v_i\}_{i=1}^n$ is an orthonormal basis for V then for any $x, y \in V$,*

$$\langle x, y \rangle = \sum_{i=1}^{n} \langle x, v_i \rangle \overline{\langle y, v_i \rangle}.$$

Proof: We know (by Theorem 9 of the notes, for instance) that

$$y = \sum_{i=1}^{n} \langle y, v_i \rangle v_i,$$

so

$$\begin{aligned}
\langle x, y \rangle &= \left\langle x, \sum_{i=1}^{n} \langle v_i, y \rangle v_i \right\rangle \\
&= \sum_{i=1}^{n} \langle x, \langle v_i, y \rangle v_i \rangle \\
&= \sum_{i=1}^{n} \langle v_i, y \rangle \langle x, v_i \rangle \\
\langle x, y \rangle &= \sum_{i=1}^{n} \langle x, v_i \rangle \overline{\langle y, v_i \rangle},
\end{aligned}$$

as asserted. □

Q8:

Claim. *If A is an $n \times n$ matrix with n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ then $\det(A) = \prod_1^n \lambda_i$ and $tr(A) = \sum_1^n \lambda_i$.*

Proof: Since A has n distinct eigenvalues, A is diagonalizable. Let Q be an invertible matrix such that $QAQ^{-1} = D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. Then $\det(A) = \det(Q^{-1}DQ) = \det(Q)\det(Q)^{-1}\det(D) = \det(D)$. Since $\det(D) = \prod_1^n \lambda_i$, the first part of the assertion follows.

For the second assertion, we use the fact that $\mathrm{tr}(ABC) = \mathrm{tr}(CAB)$ for any three $n \times n$ matrices A, B, and C (just let me know if you have any questions about this). Then $\mathrm{tr}(A) = \mathrm{tr}(Q^{-1}DQ) = \mathrm{tr}(QQ^{-1}D) = \mathrm{tr}(D)$. Since $\mathrm{tr}(D) = \sum_1^n \lambda_i$, this provides the second part of the assertion. □

Q9:

Claim. *If V is a finite-dimensional inner product space and W is a subspace of V then $(W^\perp)^\perp = W$.*

Proof: Let $\{w_1, \dots, w_k\}$ be a basis of W, which we may and do choose to be orthonormal via the Gram-Schmidt process, and let $\{w_1, \dots, w_k, v_{k+1}, \dots, v_n\}$ be an extension of this basis to an orthonormal basis of V. Given $x \in V$, write $x = \sum_1^k \alpha_i w_i + \sum_{k+1}^n \beta_j v_j$. Then $x \in W^\perp$ iff for every w_ℓ, $1 \le \ell \le k$,

$$\begin{aligned} \langle w_\ell, x \rangle &= \sum_{j=k+1}^n \alpha_i \langle w_\ell, w_i \rangle + \sum_{j=k+1}^n \beta_j \langle w_\ell, v_j \rangle \\ &= \sum_{j=k+1}^n \alpha_i \delta_{\ell i} \\ \langle w_\ell, x \rangle &= \alpha_\ell = 0, \end{aligned}$$

since $\langle w_\ell, v_j \rangle = 0$ for every j, so we must have $\alpha_i = 0$ for every i. Thus $x \in W^\perp$ iff x has the form

$$x = \sum_{j=k+1}^n \beta_j v_j,$$

i.e. $W^\perp = \mathrm{span}(v_{k+1}, \dots, v_n)$. If we then reverse this argument, we find that $x \in (W^\perp)^\perp$ iff $\beta_j = 0$ for every j, i.e. $x \in \mathrm{span}(w_1, \dots, w_k) = W$. □

Remark: This also holds in complete inner product spaces of infinite dimension (known as Hilbert spaces), but the arguments involved are quite a bit more sophisticated.