(Partial) Solutions to Homework 7

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Q6:

Definition. *A scalar matrix is a square matrix of the form λI for some scalar λ, i.e. a scalar matrix is a diagonal matrix in which all the diagonal entries are equal.*

Claim. *1. If a square matrix A is similar to a scalar matrix λI then $A = \lambda I$.*

2. A diagonal matrix having only one eigenvalue is a scalar matrix.

3. The matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

Proof: 1. Suppose that A is similar to λI for some scalar λ, i.e. that $QAQ^{-1} = \lambda I$ for some λ. Then $A = \lambda Q^{-1} I Q = \lambda I$, which is the assertion.

2. If A is a diagonal matrix then certainly the elements of the standard basis, $\{e_1, \ldots, e_n\}$ are all eigenvalues of A, namely $Ae_i = \lambda_i e_i$, where λ_i is the ith diagonal entry. Since A has only one eigenvalue, it follows that every λ_i must be equal. If we denote this scalar by λ, then we find that $A = \lambda I$.

3. The characteristic equation of this matrix is given by $(\lambda - 1)^2 = 0$, so our matrix has only one eigenvalue. If the matrix were diagonalizable, then it would be a scalar matrix by (2). Since it is clearly not, it follows that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable. □

Q7:

Claim. *Let T be a linear operator on a vector space V, and let x be an eigenvector of T corresponding to the eigenvalue λ. If $m \in \mathbb{N}$ then x is an eigenvector of T^m corresponding to the eigenvalue λ^m.*

Proof: By induction. The case $n = 1$ is true by assumption. Now suppose the assertion holds for $n = m$. We remark that T^m, T^{m+1} are linear maps. (This can be proven by induction: the composition of finitely many linear maps is linear. Let me know if there are any questions about this.) Now $T^{m+1}x = T^m(Tx) = T^m(\lambda x) = \lambda T^m x = \lambda^{m+1} x$, where we have used the inductive assumption in the last equality. The claim then follows from induction. □

Q9:

Claim. *If A and B are similar $n \times n$ matrices then A and B have the same set of eigenvalues.*

Proof: Suppose $QAQ^{-1} = B$. Then

$$\begin{aligned} \det(B - \lambda I) &= \det(QAQ^{-1} - \lambda QQ^{-1}) \\ &= \det(Q(A - \lambda I)Q^{-1}) \\ &= \det Q \det(Q^{-1}) \det(A - \lambda I) \\ \det(B - \lambda I) &= \det(A - \lambda I) \end{aligned}$$

so that the characteristic polynomials are the same. Note that we have used the fact that $\det(Q^{-1}) = \det(Q)^{-1}$. □

Q10:

Claim. *Let θ be a real number, and let A be the 2×2 rotation matrix*

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

1. *The matrix A has eigenvalues $e^{\pm i\theta}$.*
2. *The matrix A may be written as $A = QDQ^{-1}$ for an invertible matrix Q and a diagonal matrix D. Specifically, we may take*

$$\begin{aligned} D &= \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \\ Q &= \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \end{aligned}$$

3. *If $n \in \mathbb{N}$ then*

$$A^n = \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix}.$$

Proof: 1. Here A has the characteristic equation

$$(\cos\theta - \lambda)^2 + \sin^2\theta = 0,$$

from which we obtain eigenvalues

$$\begin{aligned} (\cos\theta - \lambda)^2 &= -\sin^2\theta \\ \cos\theta - \lambda &= \pm i \sin\theta \end{aligned}$$

$$\lambda = \cos\theta \pm i\sin\theta = e^{\pm i\theta},$$

as claimed.

2. The idea here is to transform our space into a basis of eigenvectors for A. Certainly A will be diagonal under this basis. To find the eigenvectors, we must solve the systems $(A - e^{\pm i\theta})\vec{v}_{\pm} = 0$, which generates the systems of equations

$$\left\{ \begin{array}{rrcl} \mp i\sin\theta x_{\pm} & -\sin\theta y_{\pm} & = & 0 \\ \sin\theta x_{\pm} & \mp i \sin\theta y_{\pm} & = & 0 \end{array} \right. ,$$

where we have taken $\vec{v}_\pm = (x_\pm, y_\pm)$. It turns out that one set of solutions for these equations are $\vec{v}_+ = (1, i)$ and $\vec{v}_- = (i, 1)$. To obtain this, we note that in each set of equations we have the freedom to choose the first coordinate of the vector (the details of solving these equations is neglected here in the interest of brevity). Thus we consider the change of coordinates matrices

$$Q = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

$$Q^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}.$$

Then by performing a calculation, we find that

$$\begin{aligned}
AQ &= \begin{pmatrix} i\cos\theta - \sin\theta & \cos\theta - i\sin\theta \\ \cos\theta + i\sin\theta & i\cos\theta - \sin\theta \end{pmatrix} \\
&= \begin{pmatrix} ie^{-i\theta} & e^{-i\theta} \\ e^{i\theta} & ie^{i\theta} \end{pmatrix} \\
&= \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \\
AQ &= QD,
\end{aligned}$$

where we have used that multiplying a matrix by a diagonal matrix on the right is the same as multiplying each column by the respective diagonal element. Certainly the above equation is equivalent to the claim.

3. We note firstly that $A^n = QD^nQ^{-1}$, which we prove by induction. The base case, $n = 1$ is clearly true. If this equation holds for $n = m$ then $A^{m+1} = AA^m = QDQ^{-1}QD^mQ^{-1} = QD^mQ^{-1}$, so that the equation holds for $n = m + 1$. That this equation always holds follows from induction.

Certainly

$$D^n = \begin{pmatrix} e^{-in\theta} & 0 \\ 0 & e^{in\theta} \end{pmatrix}.$$

Then we have that

$$\begin{aligned}
A^n &= \frac{1}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} e^{-in\theta} & 0 \\ 0 & e^{in\theta} \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} e^{-in\theta} & -ie^{-in\theta} \\ -ie^{in\theta} & e^{in\theta} \end{pmatrix} \\
&= \frac{1}{2}\begin{pmatrix} e^{in\theta} + e^{-in\theta} & i(e^{in\theta} - e^{-in\theta}) \\ -i(e^{in\theta} - e^{-in\theta}) & e^{in\theta} + e^{-in\theta} \end{pmatrix} \\
A^n &= \begin{pmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{pmatrix},
\end{aligned}$$

as asserted. $\square$