Abstract. Energy heuristics predict global existence from small data for semi-linear Klein-Gordon systems when the non-linearity has order $p > 1 + \frac{2}{n}$. We complement these heuristics and previous rigorous existence results by demonstrating blow-up from small data for $p \leq 1 + \frac{2}{n}$.

SMALL DATA BLOWUP FOR SEMILINEAR KLEIN-GORDON EQUATIONS

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1. Introduction

Fix $n \geq 0$, and let $\mathbf{R}^{n+1}$ be Minkowski space-time with the metric $ds^2 = dx^2 - dt^2$ and d'Alembertian[1]

$$\Box = (-\frac{\partial^2}{\partial t^2} + \Delta).$$

We consider the global Cauchy problem for nonlinear Klein-Gordon equations with small Cauchy data at some given time t_0,

$$\begin{cases} (\Box - 1)u = F(u, \partial_t u, \nabla_x u), & t > t_0 \\ u(t_0, \cdot) = \varepsilon f, \ \partial_t u(t_0, \cdot) = \varepsilon g \end{cases} \tag{1}$$

Here u is either a scalar field (i.e. real-valued) or a system of scalar fields, f, g are smooth functions supported on the unit ball, $\varepsilon > 0$ is a small parameter, and the non-linearity $F = F_p$ satisfies[2]

$$\begin{cases} |\partial^\alpha F(u)| \lesssim |u|^{p-|\alpha|}, & 0 \leq |\alpha| \leq \lfloor p \rfloor \\ |\partial^\alpha F(u) - \partial^\alpha F(v)| \lesssim |u - v|^{p-|\alpha|}, & p - 1 < |\alpha| < p \end{cases} \tag{2}$$

for some $p > 1$. Here $\lfloor p \rfloor$ is the largest integer less than or equal to p. When $F = F(u, \partial_t u, \nabla_x u)$, (2) is modified in the obvious way - replacing u, v with the vector representing the arguments of F. In what follows, we will distinguish between the 'semilinear' nonlinearity $F = F(u)$ and the 'semilinear with derivatives' $F = F(u, \partial_t u, \nabla u)$. This terminology is not universal.

By time translation invariance the choice of t_0 does not affect the well-posedness of this problem. If $t_0 > 1$, then by finite speed of propagation any regular solution u to (1) must be supported in the forward light-cone $\Gamma_+ = \{|x| \leq t\}$.

A simple heuristic for the relationship between the power p and global regularity comes from the energy method, which is based on an estimate of the form

$$\begin{aligned} \|u(T)\|_{H^s} &\lesssim \varepsilon + \int_{t<T} \|F(u)\|_{H^{s-1}} \, dt \\ &\lesssim \varepsilon + \int_{t<T} \|u\|_\infty^{p-1} \, dt \sup_{t<T} \|u(T)\|_{H^s}. \end{aligned}$$

[1]Some authors prefer to reverse the sign of the metric, so that the Klein-Gordon operator becomes $\Box + 1$.

[2]Examples to have in mind are $F(u) = \pm|u|^{p-1}u$ or $F(u_t) = \pm|u_t|^{p-1}u_t$.

Thus, in order to close the argument, one presumably needs $||u||_\infty^{p-1}$ to be globally integrable. Since the linear solutions to the Klein-Gordon equation decay like $t^{-n/2}$, one obtains the heuristic constraint $\frac{n}{2}(p-1) > 1$, i.e.

$$p > 1 + \frac{2}{n}. \tag{3}$$

The same heuristic can be followed for semilinear F involving derivatives. Before presenting our results, we review how this heuristic has held up under previous investigations.

We disuss first the semilinear problem with derivatives $F = F(u, \partial_t u, \nabla_x u)$. For dimensions $n \geq 3$, and smooth nonlinearity at least quadratic, global regularity of small smooth solutions was proven using distinct techniques by Shatah [26] and Klainerman [20]. These results were extended to dimension $n = 2$ in [24], [30], which according to the heuristic condition (3) would be sharp. For smooth cubic non-linearities one has global regularity for $n > 1$, and almost global existence for $n = 1$, while one has global existence for any higher order smooth non-linearity for all $n \geq 1$. (Hörmander [14]. Note that this result is also in line with the condition (3).) For a restricted class of quadratic nonlinearities in $n = 1$, existence up to time ϵ^{-4} is proven in [8].

One aim of this paper is to complement the above existence results with a blow-up theorem applicable to all dimensions. For simplicity[3] we restrict ourselves to solutions with finite energy; that is, we assume that

$$u \in C(H^1) \cap C^1(L^2).$$

We will also assume finite speed of propagation, so that u is supported in the forward light cone Γ_+.

Theorem 1.1. *If $n \geq 0$ and $1 < p \leq 1 + \frac{2}{n}$, then there exists a scalar equation (1) with the non-linearity $F(u, \partial_t u, \nabla_x u)$ satisfying (2), which does not have global finite-energy solution in Γ_+ for any non-zero smooth compactly supported data (f, g) and $\varepsilon > 0$.*

See (78), (79) below for the choice of F in Theorem 1.1. The blowup argument also gives estimates for the lifespan of the solution (see (91), (92) below).

These lifespan estimates complement previous work. For example, in the case $p = 3, n = 1$, our nonlinearity is smooth and we show that the lifespan $T_\epsilon = O(e^{\frac{C}{\epsilon^2}})$. This answers a conjecture posed in [14] where a lower bound of the same order is proven. We have learned that [41] also contains a proof of this sharp upper bound.

[3] One might ask whether the solution can be continued in a weaker norm than the natural energy space.

For the semilinear problem $F = F(u)$, it is by now well known that the estimates of Strichartz [37] give global well-posedness, asymptotic completeness, and scattering for small $\dot{H}^{1/2} \times \dot{H}^{-1/2}$ data if $1 + \frac{4}{n} < p < 1 + \frac{4}{n-1}$ (see e.g. Strauss [34]). By applying the arguments of Cazenave and Weissler [7] (or [21]) to the Strichartz estimate mentioned above, one has analogues of this result for higher powers p.

The local theory (e.g. [35], or more recently [31]) gives that (1) is locally well-posed in H^1 when $1 < p < 1 + \frac{4}{n-2}$. Thus one way to obtain a global solution for this range of p is to find uniform H^1 bounds. This is possible if (1) is Hamiltonian; that is, if there exists a scalar potential $V(u)$ such that $\nabla V(u) = F(u)$. We may of course set $V(0) = 0$. For such systems we have the conserved quantity

$$H(0) \;=\; H(t) \;=\; \int \frac{1}{2}|u_t|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u|^2 + V(u)\; dx \tag{4}$$

and so if V is positive (or satisfies a coercivity condition; see Cazenave [6]) one has global existence even for large H^1 data[4].

If V is not coercive then one cannot hope for global existence for *large* data, since blowup examples are easily gathered from blowup in the 0-dimensional problem. However, by an argument found in Cazenave [6] one still has global existence for *small* H^1 data, wherein the non-linear part of the energy $\int V(u)\; dx$ can be controlled by the linear part. More precisely, we have from Hölder's inequality and Sobolev embedding that

$$\begin{aligned} \left|\int V(u)\; dx\right| &\lesssim \|u\|_{p+1}^{p+1} \\ &\lesssim (\|u\|_2^{\theta}\|u\|_{\frac{2n}{n-2}}^{1-\theta})^{p+1} \\ &\lesssim \|u\|_{H^1}^{p+1} \end{aligned}$$

for some $0 < \theta < 1$. Hence (4) gives

$$\begin{aligned} \|u\|_{H^1}^2(t) &\lesssim H(t) - \int V(u)dx \\ &\lesssim \epsilon + \|u\|_{H^1}^{p+1}(t). \end{aligned}$$

For sufficiently small data, one immediately gets a uniform H^1 bound, and thus global existence for all $p > 1$. In particular one has global existence for any scalar semilinear Klein-Gordon equation, since every scalar non-linearity F has a primitive V. This is in stark contrast to the situation of zero rest mass (i.e., systems of nonlinear wave equations) where one has blow-up for small power nonlinearities. See John [16], Glassey [11], Sideris [29], [17].

It remains to discuss the small data global existence problem for non-Hamiltonian systems and powers $1 < p \leq 1 + \frac{4}{n}$. By using the fixed-time estimates of Marshall,

[4] In fact, this large data global existence result follows from a more involved argument for H^1 critical nonlinearities i.e. $p = 1 + \frac{4}{n-2}$, ; see [13], [27], [19].

Strauss, Wainger [23] and Brenner [3], Strauss [36] and Pecher [25] proved global existence and scattering in $L_t^\infty L_x^{p+1}$ for power nonlinearities in the range

$$\gamma(n) \quad < \quad p \quad \leq \quad 1+\frac{4}{n}$$

where $\gamma(n)$ is positive root of the quadratic

$$\frac{n}{2}\gamma(\gamma-1)=\gamma+1 \quad \Rightarrow \gamma(n)=\frac{n+2+\sqrt{n^2+12n+4}}{2n}. \tag{5}$$

Note that

$$1+\frac{2}{n}<\gamma(n)<1+\frac{4}{n}.$$

As mentioned in the above discussion of the semilinear problem with derivatives, energy methods give sharp results to the semilinear problem in dimensions n=1, n=2. Recently, Lindblad-Sogge [22] extended these heuristically sharp $n=1,2$ results to dimension $n=3$, proving that the semilinear system (1) admits a global solution if $p>1+\frac{2}{n}$ and $n\leq 3$. For $p=1+\frac{2}{n}$ one has almost global existence. The restriction $n\leq 3$ arises from the need to impose roughly $n/2$ orders of regularity on F. This is only reasonable if $p\geq n/2$ or p is an integer, and for $n\geq 4$ this restriction prohibits one from doing better than the results stated previously. Indeed, it is likely that one has blow-up for $p=1+\frac{2}{n}+\epsilon$ for sufficiently high n.

The second aim of this paper is to complement the above semilinear existence results with a blowup result for small data. We show here that the power $1+\frac{2}{n}$ is sharp, even for radial data. In his PhD thesis, Glassey [12] showed that for a power nonlinearity with $1<p\leq 1+\frac{2}{n}$ and rather general initial data, there is no scattering theory for (1) even for the scalar problem. However, there are no previous semilinear blowup results for small-data in $n>0$.

Theorem 1.2. *If $n=0,1,2,3$ and $1<p<1+\frac{2}{n}$, then there exists a (non-Hamiltonian) non-linearity $F(u)$ satisfying (2), such that the system (1) does not have a global distributional solution supported in Γ_+ for any non-trivial smooth data f, g and any $\varepsilon>0$.*

The system we consider is given by equations (7), (15) below. ¿From the local theory, H^1 solutions of the problem (1) are guaranteed to be supported in Γ_+. We deduce that the H^1 local solution to the semilinear problem (1) cannot be continued globally, and that the H^1 norm blows up in finite time. The argument also yields lifespan estimates, see Proposition 9.2 below.

It is worthwile to compare the regularity and blowup results for a non-Hamiltonian semilinear Klein-Gordon system (1) with corresponding results for non-linear wave equations. In this latter, zero mass case, it has been shown that the system enjoys global existence for $p>\gamma(n-1)$, $\gamma(n)$ as in (5). See [16] for $n=3$, [10] for $n=2$, and [9] for general n and further references. If $p < \gamma(n-1)$, blow-up has been established. See [29] for $n \geq 4$, [16] for n=3, [11] for $n = 2$, and [17], [28] for $n = 1$.

For dimensions $n = 1, 2, 3$, the number $1 + \frac{2}{n}$ therefore plays the role which $\gamma(n-1)$ played for wave equations. Indeed, Theorem 1.2 is inspired by [22] and the blowup results of [11], [16].

Note that in the case of the wave equation, the fundamental solution is positive in dimensions $n = 2, 3$, a fact central to the arguments of [11], [16] and (indirectly) [29]. For the Klein-Gordon equation, we do not have such positivity. Another moral difference between the wave and Klein-Gordon problems can be seen in zero-dimensions, where (1) takes the form of a forced harmonic oscillator. As we make clear below, a good deal of intuition for the PDE blowup comes from looking at the ODE problem.

2. The Kato mechanism

In this section we demonstrate a mechanism which converts a system of Klein Gordon equations with polynomial in time growth into a system with finite time blow up. We will in fact prove that a fairly dramatic blowup occurs. The mechanism is applied in section 9 to prove blowup for the semilinear problem (1). For completeness, we state here the notion of distribution solution used in Lemma 2.2.

Write $\mathcal{E}'(\mathbf{R}^N)$ for the dual of $C_0^\infty(\mathbf{R}^N)$.

Definition 2.1. The function $u \in \mathcal{E}'(\mathbf{R}^{n+1})$ satisfies (1) in the sense of distributions if $F(u) \in L^1_{\mathrm{loc}}(\mathbf{R}^{n+1})$ and for all $\phi \in C_0^\infty(\mathbf{R}^{n+1})$,

$$\begin{aligned}\int_{t>0}\int \phi F(u)\,dx dt \;&=\; \int_{t>0}\int u \,\Box\, \phi\, dx dt \\ + \int_{\mathbf{R}^n} \partial_t\phi(0,x) f(x)dx &- \int_{\mathbf{R}^n} \phi(0,x)g(x)dx\end{aligned}$$

Lemma 2.2. *If (1) is a semilinear Klein-Gordon system with a global solution u that obeys*

$$\limsup_{T\to\infty} \frac{1}{T^{n+1}} \int_{t<T} |u|^p \; dx dt = +\infty \tag{6}$$

for arbitrarily small $\varepsilon > 0$, then there is no global in time distributional solution

(u, w) supported in Γ_+ of the coupled system

$$(7) \qquad \begin{cases} (\Box - 1)u &= F(u) \\ (\Box - 1)w &= |w|^p + |u|^p \\ u(t_0, \cdot) &= \varepsilon f,\ \partial_t u(t_0, \cdot) = \varepsilon g \\ w(t_0, \cdot) &= \varepsilon \tilde{f},\ \partial_t w(t_0, \cdot) = \varepsilon \tilde{g}. \end{cases}$$

Here $f, g, \tilde{f}$ and $\tilde{g}$ are arbitrary C_0^∞ functions.

Proof

Suppose for contradiction that the right hand side of (7) is in $L^1_{\text{loc}}(\mathbf{R}^{n+1})$. Hölder's inequality then implies that the the mass functional

$$M(t) = -\int_{|x|<t} w(t, x)\ dx$$

is locally in L^p.

Integrating (7) in space one obtains

$$(8) \qquad M''(t) = -M(t) + \int_{|x|<t} |w|^p\ dx + \int_{|x|<t} |u|^p\ dx.$$

in the sense of distributions[5]. However, from Hölder's inequality we have

$$|M(t)| \le Ct^{n/p'}(\int_{|x|<t} |w|^p\ dx)^{1/p},$$

so that

$$\int_{|x|<t} |w|^p\ dx \ge \frac{|M(t)|^p}{Ct^{n(p-1)}}.$$

Combining this with (8) gives

$$(9) \qquad M''(t) \ge \frac{1}{Ct^{n(p-1)}}|M(t)|^p - M(t) + \int_{|x|<t} |u|^p\ dx.$$

We will show that this ODE blows up in finite time, thus obtaining the desired contradiction.

By elementary calculus we have

$$\inf_{a \ge 0} \frac{1}{Ct^{n(p-1)}}|a|^p - a = -Ct^n.$$

Applying this with $a = M(t)$ to (9) gives

$$M''(t) \ge \int_{|x|<t} |u|^p\ dx - Ct^n.$$

[5] In what follows, we supress the the necessary phrase "in the sense of distributions" from every displayed formula.

Integrating this twice and using the boundedness of $M(t_0)$, $M'(t_0)$ we obtain for large T,

$$M'(T) \geq \int_{t_0<t<T} \int |u|^p \, dxdt - CT^{n+1} \tag{10}$$

$$\begin{aligned} M(T) &\geq \int_{t_0<t<T} \int (T-t)|u|^p \, dxdt - CT^{n+2} \\ &\geq \frac{T}{2} \int_{t_0<t<\frac{T}{2}} |u|^p \, dxdt - CT^{n+2}. \end{aligned} \tag{11}$$

Let A be a large number to be chosen later. Using (6) we can find a large T_0 so that $\int_{t_0<t<\frac{T_0}{2}} \int |u|^p \, dxdt \gg T_0^{n+1}$. Hence (10), (11) give

$$M(T) \geq AT^n, \quad M'(T) \geq 0 \quad \forall T \in [T_0, 2T_0]. \tag{12}$$

Let T_* be the supremum of all times $T \in [T_0, 2T_0]$ such that

$$M(t) \geq AT_0^n(2T_0 - t)^{-\frac{2}{p-1}} \tag{13}$$

for all $T_0 \leq t \leq T$. From (12) we see that T_* exists and $T_* \geq T_0$. Since the right-hand side of (13) is not in L^p_{loc} we must have $T_* < 2T_0$. We will show that

$$M(T_*) > 2AT_0^n(2T_0 - T_*)^{-\frac{2}{p-1}}, \tag{14}$$

which by monotonicity implies that (13) holds for a neighborhood of T_*, contradicting the definition of T_*.

It remains to show (14). From (12) we see that M is nondecreasing in $[T_0, T_*]$, so that

$$M(t) \geq M(T_0) \geq AT_0^n \text{ for all } T_0 \leq t \leq T_*.$$

Thus, if A is sufficiently large, the first term on the right-hand side of (9) dominates the second. Discarding the non-negative third term, we have

$$M''(t) \gtrsim T_0^{-n(p-1)}|M(t)|^p,$$

which by (13) becomes

$$M''(t) \gtrsim A^p T_0^n (2T_0 - t)^{-\frac{2p}{p-1}},$$

and (14) follows (if A is sufficiently large) by integrating this estimate twice and using (12). This finishes the proof. ∎

Remarks.

The basic idea of the above mechanism goes back to Kato [17], and is very general. Informally, it states that once a PDE generates sufficiently large solutions, it can be coupled to another hyperbolic equation with a positive non-linearity to create solutions which blow up in finite time. This can be understood intuitively by considering the ODE analogue, namely

$$-w_{tt} = w + |w|^p + |u|^p, \quad w(0) = w_t(0) = 0.$$

This is the equation of a one-dimensional particle in a potential well $V(w) = \frac{w^2}{2} + \frac{|w|^p w}{p+1}$ with a forcing term $-|u|^p$. If the forcing term is too weak, then the particle remains in the well and a solution exists globally, however if the forcing term is strong enough then the particle escapes to negative infinity in finite time regardless of the initial data. (See Figure 2).

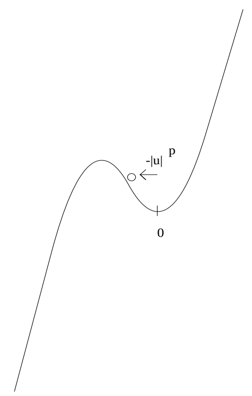

FIGURE 1. Schematic depiction of the potential well for the particle w.

An examination of the proof gives a more quantitative estimate for the time of existence:

Corollary 2.3. *If T is a large time such that*

$$\int_{t_0<t<T} \int |u|^p \, dxdt > CT^{n+1},$$

where C is a large constant, then the maximal time of existence T_* of (7) is no greater than a constant multiple of T.

3. Overview of strategy for Theorem 1.2

In this section we motivate the strategy we will use to prove Theorem 1.2, by first considering the 0-dimensional case.

¿From Lemma 2.2 we see that to prove Theorem 1.2 it suffices to find a system (1) whose solutions obey (6) for all small $\varepsilon > 0$.

The idea will be to use an iteration scheme and obtain precise control on the iterates; the hypothesis $p < 1 + \frac{2}{n}$ will be used to ensure that each iterate grows faster than its predecessor. The system we will use is of the form

$$\begin{cases} (\Box - 1)u_0 & = 0 \\ (\Box - 1)u_k & = F(u_{k-1}) \quad 1 \leq k \leq M \\ u_k(t_0, \cdot) & = \varepsilon f_k \qquad\quad\; 0 \leq k \leq M \\ \partial_t u_k(t_0, \cdot) & = \varepsilon g_k \qquad\quad\; 1 \leq k \leq M, \end{cases} \tag{15}$$

where F obeys (2). Here $u = (u_0, u_1, \ldots, u_M)$ is a system of scalar fields, $M > 0$ is an integer; these parameters will be determined later. The model for our nonlinearity is $F(u) = |u|^{p-1}u$.

Eventually (in section 9) we will prove

Theorem 3.1. *Let $n = 0, 1, 2, 3$ and $1 < p < 1 + \frac{2}{n}$. Then there exists a nonlinearity F and an integer M such that the solution to* (15) *obeys* (6) *for all small $\varepsilon > 0$ and all smooth f, g.*

Combining Theorem 3.1 and Lemma 2.2 gives Theorem 1.2.

We perform a heuristic computation to estimate the size of the iterates u_k. It is well known that the linear solution u_0 decays like $t^{-n/2}$. Thus $F(u_0)$ decays like $t^{-np/2}$. From the energy estimate it is then plausible that u_1 decays like $t^{-np/2+1}$. Note that the hypothesis $p < 1 + \frac{2}{n}$ implies that $t^{-np/2+1}$ decays slower than $t^{-n/2}$. Applying this argument inductively, one expects $|u_k| \sim t^{\alpha_k}$, where the α_k are defined by

$$\alpha_0 = -\frac{n}{2}, \quad \alpha_k = p\alpha_{k-1} + 1. \tag{16}$$

Note that α_k is a strictly increasing sequence in k. Indeed, we have

$$\alpha_k = -\frac{n}{2} + \frac{n(1 + \frac{2}{n} - p)}{2(p-1)}(p^k - 1) \geq -\frac{n}{2}. \tag{17}$$

If the above approximation $|u_k| \sim t^{\alpha_k}$ held, the proof of Theorem 3.1 would be complete since u_k (and hence u) obeys (6) as soon as $\alpha_k > 0$.

To formalize this we have to make the above approximation more precise. It turns out that an upper bound like $|u_k| \lesssim t^{\alpha_k}$ is relatively easy to obtain from energy estimates. The difficulty is to complement this estimate with lower bounds. We shall accomplish this by finding a good approximation to u_k, which can be computed explicitly.

We illustrate this approach by proving Theorem 3.1 in the 0-dimensional case $n = 0$. In this case our non-linearity will be $F(u) = |u|^{p-1}u$. For technical reasons it is convenient to extend F to the complex numbers as $F(w) = |w|^{p-1}w$.

In the 0 dimensional case, (15) is a solvable system of forced harmonic oscillators. The u_k have the explicit solution

$$u_0(t) = \varepsilon \Re(we^{it}) \tag{18}$$

$$u_k(t) = \int_{t_0}^t \Re(ie^{i(t-s)})F(u_{k-1}(s))\ ds + O(\varepsilon), \quad 1 \leq k \leq M. \tag{19}$$

where $w \neq 0$ is a constant, complex number depending on the data f_0, g_0.

We claim that

$$u_k(t) = \varepsilon^{p^k} t^{\alpha_k}(\Re(w_k e^{it}) + o(1)) \tag{20}$$

for some $w_k \neq 0$, where α_k were defined in (16).

Clearly (20) holds for $k = 0$. Now assume inductively that (20) holds for $k - 1$. Then one has

$$F(u_{k-1}(s)) = \varepsilon^{p^k} s^{p\alpha_{k-1}}(F(\Re(w_{k-1}e^{is})) + o(1)), \tag{21}$$

so by (19)

$$u_k(t) = \varepsilon^{p^k}\Re(\int_{s<t} ie^{i(t-s)}F(\Re(w_{k-1}e^{is}))\ s^{p\alpha_{k-1}} ds + o(t^{p\alpha_{k-1}+1}),$$

where we have estimated the error using the "energy estimate"

$$|\int_{t_0}^t \Re(ie^{i(t-s)})g(s)\ ds| \lesssim \int_{t_0}^t |g(s)|\ ds. \tag{22}$$

Comparing this with (20) using (16), we see that we will be done as soon as we can show

$$\int_{t_0}^t ie^{-is}F(\Re(w_k e^{is}))s^{p\alpha_{k-1}}\ ds = t^{p\alpha_{k-1}+1}(w_{k+1} + o(1)) \tag{23}$$

for some $w_{k+1} \neq 0$.

To do this we use the following identity which is easily proved by integration by parts.

Lemma 3.2. *If G is a periodic function on $\boldsymbol{R}$ with period 2π, and $\beta > -1$, then one has*

$$\int_{t_0}^t G(s)s^\beta\ ds = \overline{G}\frac{t^{\beta+1}}{\beta+1} + H(t)t^\beta - \int_{t_0}^t H(s)\beta s^{\beta-1}\ ds - \overline{G}\frac{t_0^{\beta+1}}{\beta+1}, \tag{24}$$

for large t, where

$$\overline{G} = \frac{1}{2\pi}\int_0^{2\pi} G(s)\ ds$$

and H is the periodic function

$$H(t) = \int_{t_0}^t (G(s) - \overline{G})\ ds.$$

Apply (24) with $G(s) = ie^{-is}F(\Re(w_k e^{is}))$ and $\beta = p\alpha_{k-1}$ to (23) and use the estimates $G, H = O(1)$. We will be done when we show that $\overline{G} \neq 0$. This follows from straightforward algebra which we record here for reference,

$$\begin{aligned} 0 \neq \frac{1}{2\pi}\int_0^{2\pi} \left|\Re(w_k e^{is})\right|^{p+1} ds \\ = \frac{1}{2\pi}\int_0^{2\pi} \Re(w_k e^{is})F(\Re(w_k e^{is}))ds \\ = \frac{1}{2\pi}\Re\left(\int_0^{2\pi} w_k e^{is} F(\Re(w_k e^{is}))ds\right) \\ = \frac{1}{2\pi}\Re\left(\int_0^{2\pi} \overline{w}_k e^{-is} F(\Re(w_k e^{is}))ds\right) \end{aligned}$$

from which $0 \neq \overline{G}$ follows.

Thus (20) holds for all k. ¿From this we see (6) whenever $\alpha_k > 0$. By the previous discussion this concludes the proof of Theorem 1.2, when $n = 0$.

¿From the above arguments we observe two qualitative phenomena. First, the fields u_k that solve (15) tend to be "resonant" in the sense that they contain an oscillation of the form e^{it}; this resonance is preserved under the iteration scheme. Furthermore, if such a resonant function is placed in the non-linearity, then the next iterate will be quite large; indeed, for a large proportion of times t we have the reverse energy inequality

$$|u_k(t)| \gtrsim \int_{t_0}^t |F(u_{k-1}(s))|\ ds.$$

It is the existence of this reverse energy inequality that allows one to reverse the heuristic estimate $|u_k| \lesssim t^{\alpha_k}$ mentioned earlier.

The proof in higher-dimensions will be a modification of the above arguments, though of course many further issues and complications will arise. We end this section by discussing in general terms how each of the steps in the above arguments are extended to $n = 1, 2, 3$.

For technical reasons we need F to have more than $\frac{n}{2}$ degrees of differentiability. Accordingly, we will assume for the moment that $p > \frac{n}{2}$, so that one can choose $F(u) = |u|^{p-1}u$ as in the 0-dimensional case. This assumption will be removed in Section 9.

The first step is to find an analogue of (18). Fortunately the asymptotics for the free Klein-Gordon equation are well-studied, and we can simply invoke the results in [14]. In particular, we will obtain the approximation

$$u_0 \sim \varepsilon \rho^{-\frac{n}{2}} \Re(w e^{i\rho})$$

where $\rho = (t^2 - |x|^2)^{1/2}$, w is a homogeneous symbol of order 0, and the error term is smaller than $\rho^{-\frac{n}{2}}$ in a certain sense to be made precise later.

Thus u_0 does not oscillate as a function of t, but rather as a function of ρ. Because of this and similar considerations, we will need to perform our calculations in hyperbolic polar co-ordinates $(t, x) = \rho\omega$, where ω lies on the unit hyperboloid

$$\mathbf{H}^n = \{(t, x) \in \Gamma_+ : t^2 - x^2 = 1\}. \tag{25}$$

This necessitates a certain number of preliminaries which we cover in Section 5. Also, for technical reasons it is convenient to move the Cauchy data from the constant-time surface $t = t_0$ to a hyperboloid $\rho = \rho_0$; this will be done using the local theory in Section 8.

To mimic the 0-dimensional arguments we need to find a substitute for (20). The first thing to do is to decide how to control the error terms rigorously.

In the classical energy method one controls the solution and any error terms using the Sobolev space H^s, where s is some number greater than $\frac{n}{2}$. It is therefore natural to control these errors in the hyperbolic version $H^s(\mathbf{H}^n)$ of these spaces. However, as the non-linearity F is only p times differentiable we will also need $s < p$. Thus s will most likely not be an integer.

The theory of fractional Sobolev spaces on $\mathbf{H}^n$ is technical and not well developed in the literature[6]. To avoid these complications, we will not use these spaces, but instead rely on the Hölder spaces $\Lambda^2_s(\mathbf{H}^n)$ as an acceptable substitute; this follows the approach in [22]. These spaces are of comparable strength to the Sobolev spaces, but are more elementary and thus easier to work with. We review some relevant properties of Euclidean Hölder spaces in Section 4, and discuss the hyperbolic analogues in Section 5. We will also introduce some analogues of big-O and little-o notation for these Hölder spaces, so that we can control the error terms easily.

To adapt the inductive argument used to prove (20) we need three tools. First, we need a form of Schauder's lemma, which states roughly that the map $u \to F(u)$ maps Λ^2_s to itself with a certain amount of Hölder continuity, for $s > \frac{n}{2}$. This type of result is needed to derive (21) from (20). This lemma will be proven in Sections 4, 5. Second, we need a hyperbolic energy estimate to replace (22), using the Hölder space norms Λ^2_s. For integer s versions of these estimates are well-known (e.g. [14]); the fractional counterpart is deduced easily by the symmetries of the Klein-Gordon equation. We develop the fractional analogue in Section 7. Finally, we need a substitute for the explicit formula (19). This will be accomplished by approximating the n-dimensional Klein-Gordon equation by the 0-dimensional

[6]For recent use of these spaces, see [39], [15].

equation. The informal argument is as follows. If one writes the inhomogeneous Klein-Gordon equation

$$(\Box - 1)u = F$$

in polar co-ordinates and discards the angular portion of the D'Alembertian, one obtains

$$(-\frac{\partial^2}{\partial \rho^2} - \frac{n}{\rho}\frac{\partial}{\partial \rho} - 1)u \approx F.$$

Making the substitution $u = \rho^{-n/2}v$, $F = \rho^{-n/2}G$, we obtain

$$(-\frac{\partial^2}{\partial \rho^2} - 1 + \frac{n(n-2)}{4\rho^2})v \approx G.$$

Discarding[7] the $\frac{n(n-2)}{4\rho^2}$ term we obtain the forced harmonic oscillator. Solving this in terms of u and F we obtain the approximation

$$u(\rho) \approx \rho^{-\frac{n}{2}} \int_{\sigma<\rho} \Re(ie^{i(\rho-\sigma)})\sigma^{\frac{n}{2}} F(\sigma)\, d\sigma. \tag{26}$$

We present the above steps in rigorous detail in Section 9. For now, we turn to the many preliminaries indicated above.

4. Euclidean Hölder spaces

In this section we review the properties of the Hölder spaces[8] $\Lambda^q_s(\mathbf{R}^n)$ in Euclidean space. For simplicity we restrict ourselves to non-integer $s > 0$; for integer s, see [32]. For the further properties of these spaces, see [2]; see also [40], [32], [1].

Definition 4.1. [32] Suppose that $1 < q < \infty$. If $0 < s < 1$, then Λ^q_s is the completion of the Schwarz functions under the norm

$$||h||_{\Lambda^q_s} = ||h||_q + \sup_{t \in \mathbf{R}^n} \frac{||h - h_t||_q}{|t|^s},$$

where $h_t(x) = h(x+t)$ is the translation of h by t. For $s > 1$, we define the Hölder norms inductively by

$$||h||_{\Lambda^q_s} = ||h||_q + ||\nabla h||_{\Lambda^q_{s-1}}.$$

[7] One may keep this term and approximate u by Bessel functions instead, but we shall not do so here.

[8] The authors are indebted to M. Christ for simplifying many of the arguments in this and the next section.

In particular, when $1 < s < 2$ the norm is

$$\|h\|_{\Lambda^q_s} = \|h\|_q + \|\nabla h\|_q + \sup_{t \in \mathbf{R}^n} \frac{\|\nabla h - \nabla h_t\|_q}{|t|^{s-1}}, \tag{27}$$

Note that one can impose the restriction $|t| < 1$ without significantly changing the above norms.

We begin with a Sobolev embedding lemma for Hölder spaces; for a proof, see [2].

Lemma 4.2. *Let $1 < r \leq \infty$ and $1 < s < 2$. Then*

$$\|h\|_r \lesssim \|h\|_{\Lambda^q_s}$$

whenever $\frac{1}{q} - \frac{s}{n} < \frac{1}{r} \leq \frac{1}{q}$, and

$$\|\nabla h\|_r \lesssim \|h\|_{\Lambda^p_s}$$

whenever $\frac{1}{p} - \frac{s-1}{n} < \frac{1}{r} \leq \frac{1}{p}$.

Corollary 4.3. *If $qs > n$ and $1 < s < 2$, then for sufficiently small $\varepsilon > 0$ we have*

$$\|h\|_\infty \lesssim \|h\|_{\Lambda^q_s} \tag{28}$$

$$\|\nabla h\|_{qs+\varepsilon} \lesssim \|h\|_{\Lambda^q_s} \tag{29}$$

$$\|h - h_t\|_{qs+\varepsilon} \lesssim |t| \|h\|_{\Lambda^q_s}. \tag{30}$$

Proof The first two estimates are immediate from Lemma 4.2, the hypothesis $qs < n$, and the identity

$$\frac{1}{qs} = \frac{1}{q} - \frac{s-1}{qs}.$$

The last estimate comes from the second estimate, Minkowski's theorem, and the fundamental theorem of Calculus. ■

We now give a form of Schauder's lemma for Euclidean Hölder spaces.

Lemma 4.4. *Let u, v be functions in Λ^q_s for some $1 < s < 2$, $1 < q < \infty$, and suppose that F obeys (2) for some $p > s$. If $qs > n$, then*

$$\|F(u)\|_{\Lambda^q_s} \lesssim \|u\|^p_{\Lambda^q_s}.$$

Furthermore, there exists $\delta > 0$ such that

$$\|F(u) - F(v)\|_{\Lambda^q_s} \lesssim \|u - v\|^\delta_{\Lambda^q_s} (\|u\|^{p-\delta}_{\Lambda^q_s} + \|v\|^{p-\delta}_{\Lambda^q_s}).$$

Proof Unlike the corresponding argument for Sobolev spaces (see e.g. [5], [18]), the proof will be quite elementary, consisting entirely of repeated applications of Hölder's inequality, Sobolev embedding, and arithmetic manipulation.

We prove the latter inequality only, since the former follows by setting $v = 0$. We rescale so that

$$\|u\|_{\Lambda^q_s} + \|v\|_{\Lambda^q_s} \sim 1.$$

We have to show that

$$\begin{aligned} &\|F(u) - F(v)\|_q + \|F'(u)\nabla u - F'(v)\nabla v\|_q \\ &+ \sup_t \frac{\|F'(u)\nabla u - F'(v)\nabla v - F'(u)_t \nabla u_t - F'(v)_t \nabla v_t\|_q}{t^{s-1}} \lesssim \|u - v\|^\delta_{\Lambda^q_s} \end{aligned} \tag{31}$$

for some $\delta > 0$.

¿From Corollary 4.3 we see that u, v are bounded. Combining this with (2) we see that F is Lipschitz, and F' Hölder continuous of order $s - 1 + \varepsilon$ for some $\varepsilon > 0$, in the range of u and v. In particular, we record the estimates

$$|F(u) - F(v)| \lesssim |u - v| \tag{32}$$

$$|F'(u) - F'(v)| \lesssim |u - v|^{s-1+\varepsilon} \tag{33}$$

$$\lesssim \|u - v\|^{s-1+\varepsilon}_{\Lambda^q_s} \tag{34}$$

$$|F'(v) - F'(v)_t| \lesssim |v - v_t|^{s-1+\varepsilon} \tag{35}$$

The first term in (31) is controlled by (32) and (27).

to control the second term in (31) we use the identity

$$f'(u)\nabla u - f'(v)\nabla v = f'(u)\nabla(u - v) + (f'(u) - f'(v))\nabla v.$$

the first summand is controlled by the boundedness of $f'(u)$ and (27), while the second summand is controlled by (34) and (27).

Finally we have to show that

$$\|f'(u)\nabla u - f'(v)\nabla v - f'(u)_t \nabla u_t - f'(v)_t \nabla v_t\|_q \lesssim t^{s-1}\|u - v\|^\delta_{\lambda^q_s}.$$

We use the identity

$$\begin{aligned} f'(u)\nabla u - f'(v)\nabla v - f'(u)_t \nabla u_t - f'(v)_t \nabla v_t &= f'(v)_t(\nabla(u - v) - \nabla(u - v)_t) \\ &+ (f'(u) - f'(v))\nabla(u - u_t) \\ &+ (f'(v) - f'(v)_t)(\nabla(u - v)) \\ &+ (f'(u) - f'(u)_t - f'(v) + f'(v)_t)\nabla u_t \end{aligned} \tag{36}$$

and control the four terms seperately. the first term is controlled by the boundedness of $f'(v)_t$ and (27). the second term is controlled by (34) and (27). to control the third term we use (35) and hölder's inequality to give

$$\|(f'(v) - f'(v)_t)(\nabla(u-v))\|_q \lesssim \|v - v_t\|_{q(s+\varepsilon)}^{s-1+\varepsilon} \|\nabla(u-v)\|_{q(s+\varepsilon)}. \tag{37}$$

Now applying (30) and (29) gives us the desired estimate after discarding a positive power of t, which is bounded.

To control the fourth term in (36), we will show that

$$\|(f'(u) - f'(v))\nabla u_t\|_q \lesssim \|u - v\|_{\lambda_s^q}^{s-1}.$$

The other two summands are treated analogously. Apply (33) and Hölder's inequality to get,

$$\|(F'(u) - F'(v))\nabla u_t\|_q \lesssim \|u - u_t\|_{q(s+\epsilon)}^{s-1+\epsilon} \cdot \|\nabla u_t\|_{q(s+\epsilon)}$$

and the desired estimate now follows from (30) and (29) as before. ■

5. Hyperbolic space

In this section we define our notation for the hyperbolic space $\mathbf{H}^n$, and extend the Hölder spaces in the previous section to this setting.

We will always model hyperbolic space by the unit hyperboloid[9] $\mathbf{H}^n$ which we endow with the Riemannian metric induced from the ambient Minkowski metric $ds^2 = dx^2 - dt^2$. The induced measure from this metric is $d\omega = \frac{dx}{t}$. The Lorentz group $SO(n,1)$ acts isometrically on this space, and gives rise to the Lorentz vector fields $\Omega_{\alpha\beta}$ on $\mathbf{H}^n$ defined for $\alpha, \beta = 0, \dots, n$

$$\Omega_{\alpha\beta} = x_\alpha \partial_\beta - x_\beta \partial_\alpha,$$

where we have the usual conventions

$$x_i = x^i, x_0 = -t, \partial_i = \frac{\partial}{\partial x^i}, \partial_0 = \frac{\partial}{\partial t}.$$

We extend these vector fields to all of Γ_+ in the obvious way. We observe that the vector fields $\Omega_{\alpha\beta}$ span the tangent space at every point of $\mathbf{H}^n$. Indeed, we have

$$|\nabla_H u|^2 = c \sum_{\alpha\beta} |\Omega_{\alpha\beta} u|^2$$

for some constant $c > 0$, where $\nabla_H u$ denotes the gradient of u in $\mathbf{H}^n$ with respect to the induced metric.

[9] Defined in (25).

We will often parameterize Γ_+ by polar co-ordinates $(t, x) = \rho\omega$ where $\rho = \sqrt{t^2 - |x|^2}$ and $\omega \in \mathbf{H}^n$. We recall that the D'Alembertian in this co-ordinate systems is

$$\Box = -\frac{\partial^2}{\partial\rho^2} - \frac{n}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\Delta_H, \tag{38}$$

where Δ_H denotes the Laplace-Beltrami operator on $\mathbf{H}^n$ applied to the ω variable.

We now define the hyperbolic Hölder spaces $\Lambda^q_s(\mathbf{H}^n)$, following [22]. In analogy with the Euclidean case, we define theses spaces using the norm

$$||h||_{\Lambda^q_s} = ||h||_q + \sup_{\Omega\in SO(n,1)} \frac{||h - h_\Omega||_q}{||I - \Omega||^s}$$

for $0 < s < 1$ and

$$||h||_{\Lambda^q_s} = ||h||_q + ||\nabla_H h||_{\Lambda^q_{s-1}}$$

for $s > 1$. Here $h_\Omega(\omega) = h(\Omega\omega)$ is the pullback of h with respect to Ω and $||I - \Omega||$ is the distance from Ω to the identity in $SO(n, 1)$.

Note that one can impose the restriction $||I - \Omega|| \lesssim 1$ on Ω without significantly affecting the above norms. This implies the following localization property for the Λ^q_s norm. If one partitions $\mathbf{H}^n$ into a finitely overlapping family of unit balls B (as measured by the induced Riemannian metric) and constructs a partition of unity ψ_B subordinate to that family, then

$$||h||_{\Lambda^q_s} \sim (\sum_B ||h\psi_B||^q_{\Lambda^q_s})^{1/q}.$$

This allows us to reduce the study of Hölder spaces on $\mathbf{H}^n$ to Hölder spaces on unit balls in $\mathbf{H}^n$, which by Lorentz invariance we can shift to the unit ball around the origin e_0.

If B is a ball in $\mathbf{H}^n$, let $\Lambda^q_s(B)$ be the closure of the Schwarz functions on B under the $\Lambda^q_s(\mathbf{H}^n)$ norm. Similarly if B is a ball in $\mathbf{R}^n$.

Lemma 5.1. *Let B, $\tilde{B}$ are closed balls of roughly unit radius on $\mathbf{H}^n$ and $\mathbf{R}^n$ respectively, and suppose that ϕ is a diffeomorphism from B to $\tilde{B}$. Then the map $f \mapsto f \circ \phi^{-1}$ is an isomorphism from $\Lambda^q_s(B)$ to $\Lambda^q_s(\tilde{B})$.*

This is essentially the well-known fact that Hölder spaces are invariant under diffeomorphisms (see e.g. [40]). For completeness we sketch a proof below.

Proof We will show this for $0 < s < 1$; the general case is similar. By Lorentz invariance (for $\mathbf{H}^n$) and affine invariance (for $\mathbf{R}^n$) we can take B and $\tilde{B}$ to be centered at the origins e_0 and 0 respectively. Let f be a test function on B. We will show that

$$||f \circ \phi^{-1}||_{\Lambda^q_s(\tilde{B})} \lesssim ||f||_{\Lambda^q_s(B)};$$

the reverse inequality is proved similarly. We may normalize so that $||f||_{\Lambda^q_s(B)} = 1$. The estimate $||f \circ \phi^{-1}||_q \lesssim 1$ follows from the chain rule, so it suffices to show that

$$||f \circ \phi^{-1} - (f \circ \phi^{-1})_t||_q \lesssim |t|^s \tag{39}$$

for all $|t| \lesssim 1$.

For each $k \geq 0$, let ϕ_k be a non-negative bump function of height $2^{k\,dim(SO(n,1))}$ adapted to the ball

$$\{\Omega \in SO(n,1) : ||I - \Omega|| \lesssim 2^{-k}\},$$

normalized so that $\int_{SO(n,1)} \phi_k(\Omega)\ d\Omega = 1$ for all k; here $d\Omega$ is some fixed Haar measure. We adopt the convention that $\phi_{-1} = 0$. We write $f = \sum_{k\geq 0} f^k$, where

$$f^k = \int_{SO(n,1)} (\phi_k(\Omega) - \phi_{k-1}(\Omega)) f_\Omega\ d\Omega.$$

¿From the triangle inequality, (39) will follow if we can show that

$$||f^k \circ \phi^{-1} - (f^k \circ \phi^{-1})_t||_q \lesssim \min(2^{-ks}, |t| 2^k 2^{-ks})$$

for all $k \geq 0$. We show this for $k > 0$; the proof for $k = 0$ is similar. To prove the estimate for 2^{-ks} it suffices to show that

$$||f^k||_q \lesssim 2^{ks}.$$

But this follows from Minkowski's inequality, the identity

$$f^k = \int_{SO(n,1)} (\phi_k(\Omega) - \phi_{k-1}(\Omega))(f_\Omega - f)\ d\Omega$$

and the estimate

$$||f_\Omega - f||_q \lesssim 2^{-ks} \text{ when } ||I - \Omega|| \lesssim 2^{-k}. \tag{40}$$

To prove the estimate for $|t| 2^k 2^{-ks}$ it suffices by Minkowski's theorem and the fundamental theorem of Calculus to show that

$$||\nabla (f^k \circ \phi^{-1})||_q \lesssim 2^k 2^{-ks}.$$

By the chain rule and the fact that the $\Omega_{\alpha\beta}$ span the tangent space of $\mathbf{H}^n$, it thus suffices to show that

$$||\Omega_{\alpha\beta} f^k||_q \lesssim 2^k 2^{-ks}$$

for all α, β. ¿From an integration by parts on $SO(n,1)$ we see that

$$\begin{aligned} \Omega_{\alpha\beta} f^k &= \int_{SO(n,1)} (\phi_k(\Omega) - \phi_{k-1}(\Omega)) \Omega_{\alpha\beta} f_\Omega\ d\Omega \\ &= \int_{SO(n,1)} (\phi_k(\Omega) - \phi_{k-1}(\Omega)) \tilde{\Omega}_{\alpha\beta} f_\Omega\ d\Omega \\ &= \int_{SO(n,1)} (\phi_k(\Omega) - \phi_{k-1}(\Omega)) \tilde{\Omega}_{\alpha\beta} (f_\Omega - f)\ d\Omega \\ &= - \int_{SO(n,1)} \tilde{\Omega}_{\alpha\beta} (\phi_k(\Omega) - \phi_{k-1}(\Omega))(f_\Omega - f)\ d\Omega \end{aligned}$$

where $\tilde{\Omega}_{\alpha\beta}$ is the vector field on $SO(n,1)$ generated by the Lie algebra element corresponding to $\Omega_{\alpha\beta}$. The result then follows from (40), Minkowski's inequality, and the pointwise estimate

$$|\tilde{\Omega}_{\alpha\beta}(\phi_k(\Omega) - \phi_{k-1}(\Omega))| \lesssim 2^k 2^{k\,dim(SO(n,1))}.$$

■

Because of this equivalence, many properties of Hölder spaces carry over from the Euclidean setting to the hyperbolic setting. In particular, lemma 4.4 holds on $\mathbf{H}^n$ with the obvious changes.

We also define hyperbolic Sobolev spaces $H^m(\mathbf{H}^n)$ for integers $m \geq 0$ by the norm

$$\|h\|_{H^m(\mathbf{H}^n)} = \|h\|_{L^2(\mathbf{H}^n)} + \|\nabla_H^m h\|_{L^2(\mathbf{H}^n)}.$$

We will not need the non-integer Sobolev spaces, relying on the Hölder space analogues instead. Note that the H^m norm controls the Λ^2_s norm whenever $s < m$.

In order to control the error terms in the proof of Theorem 3.1, sup-norm control is not sufficient, and we must resort to Λ^2_s norms; by (28), these norms are stronger than the sup norm when $s > \frac{n}{2}$. This motivates the following definition. In the sequel $0 < \delta \ll 1$ will be a small parameter which we will choose later.

Definition 5.2. Let $1 < s < 2$ be such that $s > \frac{n}{2}$, and let $f(\rho)$ be a positive function on $\mathbf{R}^+$. Suppose that $e = e(\rho, \omega)$ is a function on Γ_+ that may depend on the parameter δ. If

$$\|e(\rho, \cdot)\|_{\Lambda^2_s(\mathbf{H}^n)} \lesssim f(\rho)$$

then we write $e = O_s(f(\rho))$. If

$$\limsup_{\rho \to \infty} \frac{\|e(\rho, \cdot)\|_{\Lambda^2_s(\mathbf{H}^n)}}{f(\rho)} \leq c(\delta)$$

for some function $c(\delta)$ that tends to zero as $\delta \to 0$, then we write $e = \tilde{o}_s(f(\rho))$.

The hyperbolic version of Lemma 4.4 can be used to control the action of the non-linearity F on expressions given by O_s and $\tilde{o}_s$ notation.

Corollary 5.3. *Suppose that s is as in Definition 5.2 and F obeys (2) for some $p > s$. Suppose that u, v are functions on Γ_+ and f is a non-negative function on $\boldsymbol{R}^+$ such that*

$$u = O_s(f(\rho)), \quad u = v + \tilde{o}_s(f(\rho)).$$

Then

$$F(u) = O_s(f(\rho)^p), \quad F(u) = F(v) + \tilde{o}_s(f(\rho)^p).$$

6. Asymptotics for the linear Klein-Gordon equation

In this section we quote some well-known asymptotics for solutions to the free Klein-Gordon equation. We will not need the full asymptotics given in [14], but content ourselves with finding the qualitative form of the leading term.

Definition 6.1. A complex-valued function w on Γ_+ is said to be a *homogeneous symbol of order 0* if $w = \phi(\frac{x}{\rho})$ for some Schwarz function ϕ.

Note that if w is a homogeneous symbol of order 0, then $w = O_s(1)$. We will often abuse notation and write $w = w(\omega)$.

Lemma 6.2. *If u solves*

$$\begin{cases} (\Box - 1)u_0 = 0, & t > t_0 \\ u_0(t_0, \cdot) = f,\ \partial_t u_0(t_0, \cdot) = g \end{cases} \tag{41}$$

for some C^∞ functions f, g supported on the unit ball which do not both vanish, then for any $1 < s < 2$ we have

$$u_0 = \rho^{-\frac{n}{2}} \Re(w e^{i\rho}) + \tilde{o}_s(\rho^{-\frac{n}{2}}) \tag{42}$$

for large ρ, where w is a homogeneous symbol of order 0 which is not identically zero.

Proof We have to estimate the error term in (42) in Λ^2_s; it will suffice to estimate it by a Sobolev space such as $H^2(\mathbf{H}^n)$. It suffices to produce $L^2(\mathbf{H}^n)$ bounds for the error, since one can recover the Sobolev bound by differentiating the equation (41) with respect to the Lorentz vector fields $\Omega_{\alpha\beta}$, and adjusting the initial data accordingly.

¿From Plancherel's theorem we see that u_0, as a function of $\mathbf{R}^{n+1}$, is the space-time Fourier transform of a non-zero smooth rapidly decreasing measure on the two-sheeted hyperboloid $\{(\tau, \xi) : \tau^2 - |\xi|^2 = 1\}$. Thus, by the usual stationary phase computations, (see e.g. [33], or [14]) we have

$$u_0 = t^{-\frac{n}{2}}(w_+ e^{i\rho} + w_- e^{-i\rho}) + O(t^{-\frac{n}{2}-1}\left(\frac{1+t-|x|}{1+t+|x|}\right)^N)$$

for arbitrarily large $N > 0$ and some homogeneous symbols w_+, w_- of order 0 which cannot both vanish. Since u is real we must have $w_+ = \overline{w_-}$, and (42) follows (with L^2 control on the error term).

A more precise version of this computation (with explicit formulae for w, etc.) may be found in [14], Chapter 7. ■

7. A hyperbolic energy estimate for Λ^2_s

Let $1 < s < 2$, $\rho_0 \gg 1$ be a large constant, and suppose that $u = u(\rho, \omega)$, $F = F(\rho, \omega)$ are given in polar co-ordinates on the upper hyperbolic region $\rho > \rho_0$. To avoid technicalities we make the a priori assumption that u and F are Schwarz functions; this assumption can be removed by the usual methods. The aim of this section is to prove the energy estimate

Proposition 7.1. *If ρ_0 is sufficiently large, $s > 0$ is a non-integer and*

$$(\Box - 1)u = F \tag{43}$$

for $\rho > \rho_0$, then

$$\begin{aligned} &\rho^{\frac{n}{2}}\left[\|\partial_\rho u(\rho,\cdot)\|_{\Lambda^2_s(\boldsymbol{H}^n)} + \|\frac{1}{\rho}\nabla_H u(\rho,\cdot)\|_{\Lambda^2_s(\boldsymbol{H}^n)} + \|u(\rho,\cdot)\|_{\Lambda^2_s(\boldsymbol{H}^n)}\right] \lesssim \\ &\rho_0^{\frac{n}{2}}\left[\|\partial_\rho u(\rho_0,\cdot)\|_{\Lambda^2_s(\boldsymbol{H}^n)} + \|\frac{1}{\rho}\nabla_H u(\rho_0,\cdot)\|_{\Lambda^2_s(\boldsymbol{H}^n)} + \|u(\rho_0,\cdot)\|_{\Lambda^2_s(\boldsymbol{H}^n)}\right] \\ &\qquad + \int_{\rho_0}^{\rho} \sigma^{\frac{n}{2}}\|F(\sigma,\cdot)\|_{\Lambda^2_s(\boldsymbol{H}^n)}\, d\sigma. \end{aligned} \tag{44}$$

This estimate essentially appears in [22] in the form of a mixed spacetime estimate; versions of it for integer numbers of derivatives appear in many places ([14], [20]).

Proof It will suffice to prove the L^2 energy estimate

$$\begin{aligned} &\rho^{\frac{n}{2}}\left[\|\partial_\rho u(\rho,\cdot)\|_{L^2(\mathbf{H}^n)} + \|\frac{1}{\rho}\nabla_H u(\rho,\cdot)\|_{L^2(\mathbf{H}^n)} + \|u(\rho,\cdot)\|_{L^2(\mathbf{H}^n)}\right] \lesssim \\ &\rho_0^{\frac{n}{2}}\left[\|\partial_\rho u(\rho_0,\cdot)\|_{L^2(\mathbf{H}^n)} + \|\frac{1}{\rho}\nabla_H u(\rho_0,\cdot)\|_{L^2(\mathbf{H}^n)} + \|u(\rho_0,\cdot)\|_{L^2(\mathbf{H}^n)}\right] \\ &\qquad + \int_{\rho_0}^{\rho} \sigma^{\frac{n}{2}}\|F(\sigma,\cdot)\|_{L^2(\mathbf{H}^n)}\, d\sigma. \end{aligned} \tag{45}$$

Indeed, since $\Box$ commutes with Lorentz transformations Ω we see that

$$(\Box - 1)\nabla^k_H u = \nabla^k_H F$$
$$(\Box - 1)\frac{\nabla^k_H u - (\nabla^k_H u)_\Omega}{\|I - \Omega\|^{s-k}} = \frac{\nabla^k_H F - (\nabla^k_H F)_\Omega}{\|I - \Omega\|^{s-k}},$$

for all $k \geq 0$ and $\Omega \in SO(n,1)$, and so (44) follows by applying (45) to each of these equations and using the definition of the $\Lambda^2_s(\mathbf{H}^n)$ norm.

It remains to prove (45). For all $\rho > \rho_0 \gg 1$, let $E(\rho) \geq 0$ be defined by

$$E(\rho)^2 = \rho^n \int_{\mathbf{H}^n} |\partial_\rho u(\rho,\omega)|^2 + \frac{1}{\rho^2}|\nabla_H u(\rho,\omega)|^2 + |u(\rho,\omega)|^2 \, d\omega + n\rho^{n-1}\int_{\mathbf{H}^n} u(\rho,\omega)\partial_\rho u(\rho,\omega)\, d\omega. \tag{46}$$

The first integral is a familiar energy density. The second integral is a correction term arising from the fact that the Klein-Gordon equation is not completely invariant under scaling. Because of this we will expect the energy identity to only be approximate.

¿From the Cauchy-Schwarz inequality we see that the second integral is $O(\frac{1}{\rho})$ of the first integral. Thus, if ρ is sufficiently large, we have

$$\rho^{\frac{n}{2}}(\|\partial_\rho u(\rho,\cdot)\|_{L^2} + \frac{1}{\rho}\|\nabla_H u(\rho,\cdot)\|_{L^2} + \|u(\rho,\cdot)\|_{L^2}) \sim E(\rho). \tag{47}$$

In particular, (45) reduces to

$$E(\rho) \lesssim E(\rho_0) + \int_{\rho_0}^{\rho} \sigma^{\frac{n}{2}}\|F(\sigma,\cdot)\|_{L^2(\mathbf{H}^n)}\, d\sigma. \tag{48}$$

Differentiating (46) with respect to ρ we obtain

$$2E(\rho)E'(\rho) =$$

$$\rho^n \int_{\mathbf{H}^n} 2\partial_\rho u\partial_\rho^2 u + \frac{2}{\rho^2}\partial_\rho\nabla_H u\nabla_H u - \frac{2}{\rho^3}|\nabla_H u|^2 + 2\partial_\rho u u + \frac{n}{\rho}(|\partial_\rho u|^2 + \frac{1}{\rho^2}|\nabla_H u|^2 + |u|^2)\, d\omega + n\rho^{n-1}\int_{\mathbf{H}^n} |\partial_\rho u|^2 + u\partial_\rho^2 u + \frac{n-1}{\rho}u\partial\rho\, d\omega.$$

Simplifying using integration by parts, this becomes

$$E(\rho)E'(\rho) = \rho^n \int_{\mathbf{H}^n}(-\frac{n}{2\rho}u - \partial_\rho u)(-\partial_\rho^2 u - \frac{n}{\rho}\partial_\rho u + \frac{1}{\rho^2}\Delta_H u - u) + \frac{n(n-1)}{2\rho^2}u\partial_\rho u - \frac{2}{\rho^3}|\nabla_H u|^2\, d\omega. \tag{49}$$

However, from (38) we have

$$-\partial_\rho^2 u - \frac{n}{\rho}\partial_\rho u + \frac{1}{\rho^2}\Delta_H u - u = -F.$$

Inserting this into the above estimate and using Cauchy-Schwarz we obtain

$$E(\rho)E'(\rho) \lesssim \rho^n[(\frac{1}{\rho}\|u\|_2 + \|\partial_\rho u\|_2)\|F(\rho)\|_2 + \frac{1}{\rho^2}\|u\|_2\|\partial_\rho u\|_2].$$

Note that we have discarded the last term in (49) since it was negative.

¿From (47) we thus obtain

$$E(\rho)E'(\rho) \lesssim (1+\frac{1}{\rho})E(\rho)\rho^{\frac{n}{2}}\|F(\rho)\|_2 + \frac{1}{\rho^2}E(\rho)^2.$$

Since we are taking ρ large, we have

$$E'(\rho) \lesssim \rho^{\frac{n}{2}}\|F(\rho)\|_2 + \frac{1}{\rho^2}E(\rho).$$

The claim (48) follows from Gronwall's inequality and the global integrability of $\frac{1}{\rho^2}$ for large ρ.

■

¿From (44) we see that there is a smoothing effect of one derivative in the ∂_ρ and $\frac{1}{\rho}\nabla_H$ directions. We will not use this smoothing, however, and content ourselves with the following weaker corollary.

Corollary 7.2. *Suppose that*

$$\|\partial_\rho u(\rho_0,\cdot)\|_{\Lambda^2_s(\boldsymbol{H}^n)} + \|u(\rho_0,\cdot)\|_{\Lambda^2_{s+1}(\boldsymbol{H}^n)} \lesssim 1.$$

If $F=0$, then $u = O_s(\rho^{-\frac{n}{2}})$. If $F = O_s(\rho^{\beta-1})$ for some $\beta > -\frac{n}{2}$, then $u = O_s(\rho^\beta)$. If $F = o_s(\rho^{\beta-1})$ for the same β, then $u = o_s(\rho^\beta)$.

8. Local existence to a hyperboloid

The objective of this section is to show that the problem (15) can be continued from the constant time slice $t = t_0$ to the hyperboloid $\rho = \rho_0$, where $\rho_0 = t_0 + 1$, and that the solution is in a Hölder space on this hyperboloid.

Proposition 8.1. *Let $n = 1,2,3$, and let $1 < s < 2$ be such that $\frac{n}{2} < s < p$. If t_0 is chosen to be sufficiently large, then*

$$\|\partial_\rho u_k(\rho_0,\cdot)\|_{\Lambda^2_s(\boldsymbol{H}^n)} + \|u_k(\rho_0,\cdot)\|_{\Lambda^2_{s+1}(\boldsymbol{H}^n)} \lesssim \varepsilon \tag{50}$$

for all $k \geq 0$. The functions u_k as in (15). The constant in (50) can depend on everything except ε).

Proof This will be a routine modification of the standard energy method techniques for non-Minkowskian spacetime.

¿From Huygens' principle we see that the support of the u_k between the constant time surface $t = t_0$ and the hyperboloid $\rho = \rho_0$ is contained in a cube Q of bounded size. Thus we may change co-ordinates $(t, x) \to (\tilde{t}, \tilde{x})$ diffeomorphically so that the surface $\{t = t_0\} \cap Q$ maps into $\{\tilde{t} = 0\}$, $\{\rho = \rho_0\} \cap Q$ maps to $\{\tilde{t} = 1\}$, and the normal vector ∂_ρ on $\{\rho = \rho_0\} \cap Q$ maps to the normal vector $\partial_{\tilde{t}}$ on $\{\tilde{t} = 1\}$. The induced metric $g^{\alpha\beta}$ on the cube will not be flat, but it can be made as close to the Minkowski metric $\eta^{\alpha\beta}$ in C^∞ as desired, if t_0 is chosen to be sufficiently large. Let $\tilde{u}_k$ denote the function u_k in these new co-ordinates.

The equation (15) is still satisfied in these new co-ordinates, with the understanding that $\Box = \Box_g$ is now the D'Alembertian with respect to the metric $g^{\alpha\beta}$. Also, the hyperbolic Hölder space norm on $\rho = \rho_0$ is equivalent (by at most a power of ρ_0) to the Euclidean Hölder space norm on $\tilde{t} = 1$. Thus (50) will follow from

$$\|\tilde{u}_k(\tilde{t}, \cdot)\|_{\Lambda^2_{s+1}(\mathbf{R}^n)} + \|\partial_{\tilde{t}} \tilde{u}_k(\tilde{t}, \cdot)\|_{\Lambda^2_s(\mathbf{R}^n)} \lesssim \varepsilon \tag{51}$$

for all $0 \leq \tilde{t} \leq 1$.

We will use the following energy estimate.

Lemma 8.2. *If $g^{\alpha\beta}$ is sufficiently close to $\eta^{\alpha\beta}$ in C^∞ norm, and*

$$\begin{cases} (\Box_g - 1)u = F, & 0 < \tilde{t} < 1 \\ u(0, \cdot) = f, \ \partial_t u(0, \cdot) = g \end{cases} \tag{52}$$

then for all $0 < \tilde{t} \leq 1$ we have

$$\begin{aligned} &\|u(\tilde{t}, \cdot)\|_{\Lambda^2_{s+1}(\boldsymbol{R}^n)} + \|\partial_{\tilde{t}} u(\tilde{t}, \cdot)\|_{\Lambda^2_s(\boldsymbol{R}^n)} \lesssim \\ &\|f\|_{\Lambda^2_{s+1}(\boldsymbol{R}^n)} + \|g\|_{\Lambda^2_s(\boldsymbol{R}^n)} + \int_0^{\tilde{t}} \|F(\tau, \cdot)\|_{\Lambda^2_s(\boldsymbol{R}^n)} \, d\tau. \end{aligned} \tag{53}$$

Proof It suffices to show the Sobolev space estimates

$$\|u(\tilde{t}, \cdot)\|_{H^{k+1}} + \|\partial_{\tilde{t}} u(\tilde{t}, \cdot)\|_{H^k} \lesssim \|f\|_{H^{k+1}} + \|g\|_{H^k} + \int_0^{\tilde{t}} \|F(\tau, \cdot)\|_{H^k} \, d\tau \tag{54}$$

for all $k \geq 0$, since the Hölder space analogues of these estimates then follow by real interpolation (see [2]). When $k = 0$, (54) is just just the energy estimate for manifolds with mild curvature (see [31], Proposition 2.1 for the wave equation; the extension to Klein-Gordon is routine.) To prove the estimate for $k > 0$, we differentiate (52) to obtain the identity

$$(\Box_g - 1)\nabla^k u = \nabla^k F + [\Box_g, \nabla^k] u$$

where ∇ is the Euclidean spatial gradient. Applying this to (54) with $k = 0$ we obtain

(55)
$$\|\nabla^k u(\tilde{t},\cdot)\|_{H^1} + \|\partial_{\tilde{t}}\nabla^k u(\tilde{t},\cdot)\|_{L^2} \lesssim \|\nabla^k f\|_{H^1} + \|\nabla^k g\|_{L^2} + \int_0^{\tilde{t}} \|\nabla^k F(\tau,\cdot)\|_{L^2} + \|[\Box_g,\nabla^k]u(s,\cdot)\|_{L^2}\, d\tau.$$

However, $[\Box_g,\nabla^k]$ is a $(k+1)^{th}$-order differential operator with bounded co-efficients. Thus we have the pointwise estimate

$$|[\Box_g,\nabla^k]u| \lesssim \sum_{i+j\leq k+1} |\nabla^i \partial_t^j u|.$$

However one can use (52) and the assumption $g^{\alpha\beta} \approx \eta^{\alpha\beta}$ to eliminate any term with two or more time derivatives. Ultimately we obtain

$$|[\Box_g,\nabla^k]u| \lesssim \sum_{i\leq k+1} |\nabla^i u| + \sum_{i\leq k} |\nabla^i \partial_t u| + \sum_{i\leq k-1} |\nabla^i F|.$$

Inserting this into (55) we obtain

$$\|u(\tilde{t},\cdot)\|_{H^{k+1}} + \|\partial_{\tilde{t}} u(\tilde{t},\cdot)\|_{H^k} \lesssim \|f\|_{H^2} + \|g\|_{H^k} + \int_0^{\tilde{t}} \|F(\tau,\cdot)\|_{H^k} + \|u(\tau,\cdot)\|_{H^{k+1}} + \|\partial_{\tilde{t}} u(\tau,\cdot)\|_{H^k}\, d\tau,$$

and (53) follows from Gronwall's inequality. ∎

¿From this lemma we see that (51) holds for $k = 0$. Assume (51) holds for $k-1$. From (15) and (51) we get

$$\|\tilde{u}_k(\tilde{t},\cdot)\|_{\Lambda^2_{s+1}(\mathbf{R}^n)} + \|\partial_{\tilde{t}}\tilde{u}_k(\tilde{t},\cdot)\|_{\Lambda^2_s(\mathbf{R}^n)} \lesssim \varepsilon + \int_0^{\tilde{t}} \|F(\tilde{u}_{k-1}(\tau,\cdot))\|_{\Lambda^2_s(\mathbf{R}^n)}\, d\tau.$$

However, from Lemma 4.4 and the induction hypothesis we have

$$\|F(\tilde{u}_{k-1}(\tau,\cdot))\|_{\Lambda^2_s(\mathbf{R}^n)} \lesssim \|\tilde{u}_{k-1}(\tau,\cdot)\|^p_{\Lambda^2_s(\mathbf{R}^n)} \lesssim \varepsilon^p \lesssim \varepsilon.$$

Thus (51) holds for all k, as desired. ∎

9. Finishing the Proof of Theorem 1.2

With these basic tools it is short work to finish the proof of Theorem 3.1 (and hence Theorem 1.2) for $n = 1,2,3$ and $1 < p < 1 + \frac{2}{n}$. Fix now p, n. We temporarily assume that $p > \frac{n}{2}$, removing this assumption at the end of the section. Fix $1 < s < 2$ such that $\frac{n}{2} < s < p$.

Fix t_0 to be any time for which Proposition 8.1 holds; it turns out that $t_0 = 2$ will suffice. We reserve M to be chosen later. We also let $0 < \delta \ll 1$ be a small number to be chosen later; the $\tilde{o}_s$ notation will depend on this parameter δ.

Our non-linearity will be $F(u) = |u|^{p-1}u$. For technical reasons, we break up the non-linearity F into a smooth function F_δ and a small function G_δ, where

$$\begin{aligned} G_\delta(u) &= F(u)\psi(\delta^{-1}u) \\ F_\delta(u) &= F(u)(1 - \psi(\delta^{-1}u)) \end{aligned} \tag{56}$$

and ψ is a bump function that equals one near the origin.

The core of the argument is the following analogue of (20).

Proposition 9.1. *For all $0 \leq k \leq M$ we have*

$$u_k(\rho, \omega) = \begin{cases} O_s(\epsilon\rho^{-\frac{n}{2}}) & \text{for } \rho \leq T_0 \\ \varepsilon^{p^k}\rho^{\alpha_k}\left(\Re(w_k^\delta(\omega)e^{i\rho}) + \tilde{o}_s(1) \right) & \text{for } \rho \geq T_0. \end{cases} \tag{57}$$

where

$$T_0 = \varepsilon^{-\frac{2(p-1)}{n(1+\frac{2}{n}-p)}} \tag{58}$$

and $w_k^\delta(\omega)$ is given recursively by

$$\begin{aligned} w_0^\delta(\omega) &= w(\omega) \\ w_k^\delta(\omega) &= \frac{1}{2\pi(\beta_k + 1)} \int_0^{2\pi} ie^{-i\sigma} F_\delta(\Re(w_{k-1}^\delta(\omega)e^{i\sigma}))\, d\sigma. \end{aligned} \tag{59}$$

Here

$$\beta_k = p\alpha_{k-1} + \frac{n}{2} = \alpha_k + \frac{n}{2} - 1 \tag{60}$$

and w,α_k are as in (42),(17) respectively.

Note that the w_k^δ are all homogeneous symbols of order 0.

Proof ¿From (42) we see that (57) holds for $k = 0$. Now assume that (57) holds for $k - 1$, where $1 \leq k \leq M$.

For $\rho \leq T_0$, (57) follows easily from the definition of T_0, Corollary 7.2, and induction. For $\rho \geq T_0$ we apply F to (57) and use Corollary 5.3 to get

$$\begin{aligned} F(u_{k-1}) &= \rho^{p\alpha_{k-1}}\varepsilon^{p^k}(F(\Re(w_{k-1}^\delta e^{i\rho})) + \tilde{o}_s(1)) \\ &= \rho^{p\alpha_{k-1}}\varepsilon^{p^k}(F_\delta(\Re(w_{k-1}^\delta e^{i\rho})) + G_\delta(\Re(w_{k-1}^\delta e^{i\rho})) + \tilde{o}_s(1)). \end{aligned} \tag{61}$$

To estimate the contribution of G_δ to (61) we observe the identity

$$G_\delta(\Re(w_{k-1}^\delta e^{i\rho})) = \delta^p G_1(\delta^{-1}\Re(w_{k-1}^\delta e^{i\rho})).$$

Let $\tilde{p}$ be any number such that $s < \tilde{p} < p$. One can easily verify that the nonlinearity G_1 obeys (2) for all $\tilde{p}$. Also, note $F_\delta(w) = O_s(1)$, and hence $w_{k-1}^\delta = O_s(1)$, where both bounds are independent of δ for all k. Combining these observations with the hyperbolic version of Lemma 4.4 we get

$$G_\delta(\Re(w_{k-1}^\delta e^{i\rho})) = \delta^p G_1(\delta^{-1}\Re(w_{k-1}^\delta e^{i\rho})) = \delta^p O_s(\delta^{-\tilde{p}}) = \tilde{o}_s(1).$$

Thus the contribution of G_δ to (61) can be absorbed into the error term.

¿From these considerations we see that the field u_k obeys the approximate differential equation

$$(\Box - 1)u_k = \rho^{p\alpha_{k-1}}\varepsilon^{p^k} F_\delta(\Re(w_{k-1}^\delta e^{i\rho})) + \tilde{o}_s(\varepsilon^{p^k}\rho^{p\alpha_{k-1}}). \tag{62}$$

¿From (26) we expect $u_k \approx \tilde{u}_k$, where

$$\tilde{u}_k = \rho^{-\frac{n}{2}}\varepsilon^{p^k}\int_{\rho_0}^{\rho} \Re(ie^{i(\rho-\sigma)})F_\delta(\Re(w_{k-1}^\delta e^{i\sigma}))\sigma^{p\alpha_{k-1}+\frac{n}{2}}\, d\sigma \tag{63}$$

More precisely, we claim that

$$u_k - \tilde{u}_k = \tilde{o}_s(\varepsilon^{p^k}\rho^{\alpha_k}). \tag{64}$$

Assuming this for the moment, the proof of (57) is now straightforward as we need only verify (57) with u_k replaced by $\tilde{u}_k$.

$$\begin{aligned}\tilde{u}_k &= \rho^{-\frac{n}{2}}\varepsilon^{p^k}\int_{\rho_0}^{\rho} \Re(ie^{i(\rho-\sigma)})F_\delta(\Re(w_{k-1}^\delta e^{i\sigma}))\sigma^{\beta_k}\, d\sigma \\ &= \rho^{-\frac{n}{2}}\varepsilon^{p^k}\Re\left(e^{i\rho}\int_{\rho_0}^{\rho} ie^{-\sigma}\cdot F_\delta(\Re(w_{k-1}^\delta e^{i\sigma}))\sigma^{\beta_k}\, d\sigma\right)\end{aligned}$$

Now applying Lemma 3.2 and using the definition of w_k^δ

$$\begin{aligned} &= \rho^{-\frac{n}{2}}\epsilon^{p^k}\Re(e^{i\rho}\rho^{\beta_k+1}w_k^\delta + \tilde{o}_s(\rho^{\beta_k+1})) \\ &= \rho^{\alpha_k}\epsilon^{p^k}(\Re(w_k^\delta e^{i\rho}) + \tilde{o}_s(1))\end{aligned}$$

as desired. Note that we could apply Lemma 3.2 here because $ie^{-i\sigma}F_\delta(\Re(w_{k-1}^\delta e^{i\sigma}))$ is smooth and rapidly decreasing in ω, and bounded and periodic in σ and because $\beta_k > -1$.

It remains to show (64). In order insure that all dependence on ϵ in (57) is explicit, we note first that the initial data's contribution to u_k can be included in the error term. This follows from (50) and a straightforward calculation which gives

$$\epsilon T_0^{-\frac{n}{2}} = \epsilon^{p^k}T_0^{\alpha_k}$$

We may therefore assume that $f_k = g_k = 0$. By Corollary 7.2 it suffices to show that

$$(\Box - 1)(u_k - \tilde{u}_k) = \tilde{o}_s(\varepsilon^{p^k}\rho^{p\alpha_{k-1}}) = \tilde{o}_s(\varepsilon^{p^k}\rho^{\alpha_k - 1}).$$

Thus we need only show that (62) holds with u_k replaced by $\tilde{u}_k$.

¿From (63) $\rho^{\frac{n}{2}}\tilde{u}_k$ is a solution to the forced harmonic oscillator; in fact,

$$(-\partial_\rho^2 - 1)(\rho^{\frac{n}{2}}\tilde{u}_k) = \rho^{p\alpha_{k-1}+\frac{n}{2}}\epsilon^{p^k}F_\delta(\Re(w_{k-1}^\delta e^{i\rho})).$$

Expanding this gives

$$(-\partial_\rho^2 - \frac{n}{\rho}\partial_\rho - \frac{n(n-2)}{4\rho^2} - 1)\tilde{u}_k = \rho^{p\alpha_{k-1}}F(\Re(w_{k-1}^\delta e^{i\rho})).$$

Combining this with (38) we obtain

$$\begin{aligned}(\Box - 1)\tilde{u}_k &= (-\partial_\rho^2 - \frac{n}{\rho}\partial_\rho + \frac{1}{\rho^2}\Delta_H - 1)\tilde{u}_k \\ &= \rho^{p\alpha_{k-1}}\epsilon^{p^k}F(\Re(w_{k-1}^\delta e^{i\rho})) + \frac{1}{\rho^2}(\Delta_H\tilde{u}_k + \frac{n(n-2)}{4}\tilde{u}_k).\end{aligned} \tag{65}$$

But from (63) and the smoothness of F_δ, w_{k-1}^δ we see that

$$\tilde{u}_k, \Delta_H\tilde{u}_k = O_s(\rho^{p\alpha_{k-1}+1}) = \tilde{o}_s(\varepsilon^{p^k}\rho^{p\alpha_{k-1}+2}).$$

Thus (65) implies (57), and Proposition 9.1 follows by induction. ∎

¿From (17) we see that $\alpha_k \to \infty$ as $k \to \infty$. Thus, we may choose M to be the integer such that

$$\alpha_M > 0 \geq \alpha_{M-1} > \ldots > \alpha_0 = -\frac{n}{2}. \tag{66}$$

¿From Proposition 9.1 we have

$$|u_M(t,x)| \gtrsim \varepsilon^{p^M}\rho^{\alpha_M}(|\Re(w_M^\delta e^{i\rho})| - \tilde{o}_s(1)).$$

¿From (28), the error term $\tilde{o}_s(1)$ can be made arbitrarily small in sup-norm by choosing δ small. Since $\alpha_M > 0$, we see that (6) will hold once we can show that $|w_M^\delta(\omega)| \gtrsim 1$ on a non-empty open set in $\mathbf{H}^n$.

Since w_M^δ converges uniformly to w_M^0 as $\delta \to 0$, it suffices to show that $w_M^0 \not\equiv 0$.

Suppose to the contrary that w_M^0 vanished identically. Using (59) and discarding non-zero factors, we thus see that

$$\int_0^{2\pi} e^{-i\sigma}F(\Re(w_{M-1}^0 e^{i\sigma}))\, d\sigma \equiv 0.$$

Multiplying this by $\overline{w_{M-1}^0}$ and taking real parts, we obtain

$$\int_0^{2\pi} \Re(\overline{w_{M-1}^0}e^{-i\sigma})F(\Re(w_{M-1}^0 e^{i\sigma}))\, d\sigma \equiv 0.$$

But this simplifies to

$$\int_0^{2\pi} |\Re(w^0_{M-1} e^{i\sigma})|^{p+1} \ d\sigma \equiv 0,$$

whence $w^0_{M-1} \equiv 0$. Iterating this argument eventually gives $w \equiv 0$, contradicting Lemma 6.2. Thus w^0_M does not vanish identically, and we are done.

This concludes the proof of Proposition 3.1 for $\frac{n}{2} < p < 1 + \frac{2}{n}$. To handle the case when $1 < p \leq \frac{n}{2}$, we first apply the above arguments to nonlinearity with power $\frac{n}{2} < p_0 < 1 + \frac{2}{n}$. ¿From (66), (57), and (28) we have

$$\|u_k\|_\infty \lesssim \varepsilon \text{ for all } 0 \leq k < M. \tag{67}$$

Thus, if ε is sufficiently small, one may replace the non-linearity $F(u) = |u|^{p_0 - 1} u$ by

$$\tilde{F}(u) = |u|^{p_0 - 1} u \psi(u)$$

for some bump function ψ such that $\psi(u) = 1$ near the origin, without affecting any of the u_k. Since $\tilde{F}$ obeys (2) for p Proposition 3.1 is thus established for this range of p.

The proof above also gives an estimate for the solution's lifespan. By Corollary 2.3, the solution will blowup at time T_* where

$$\int_{t_0 < t < T_*} \int |u|^p \ dx dt > C T_*^{n+1} \tag{68}$$

The asymptotics in (57) show we'll satisfy (68) once we have $T_* > T_0$ and

$$\epsilon^{p^k} T_*^{\alpha_k} \gtrsim C$$

Allowing k to be arbitrarily large and using (17), (58) we see that blowup occurs before time T_* satisfying

$$T_* \lesssim T_0^{1+\gamma} \tag{69}$$

where $\gamma > 0$ is an arbitrarily small number.

We obtain lower bounds for the lifespan of (1) with $F = F(u)$ by combining Proposition 7.1, the hyperbolic version of Lemma 4.4, and a variant of the classical iteration argument: by Lemma 8.1 we may assume our data $\epsilon f, \epsilon g$ is given on the hyperboloid $\rho = \rho_0$ and is small as in (50). For some $T > 0$ to be determined, define the spaces

$$X_T = \left\{ u(\rho, \omega) \mid \|u\|_{X_T} \equiv \sup_{\rho_0 \leq \rho \leq T} \rho^{\frac{n}{2}} \|u\|_{L^2(H^n)}(\rho) \lesssim \epsilon \right\} \cap Y_T \tag{70}$$

$$Y_T = \left\{ u(\rho, \omega) \mid \|u\|_{Y_T} \equiv \sup_{\rho_0 \leq \rho \leq T} \rho^{\frac{n}{2}} \|u\|_{\Lambda^2_s(H^n)}(\rho) \lesssim \epsilon \right\}. \tag{71}$$

Define the mapping S on X_T as $S(v) = u$ where

$$\begin{aligned}(\Box - 1)u &= F(v)\\ u(\rho_0, \omega) &= \epsilon f\\ \partial_\rho u(\rho_0, \omega) &= \epsilon g.\end{aligned}$$

We will argue as in e.g. [22] that when F satisfies (2) with $p < 1 + \frac{2}{n}$, and $T \lesssim T_0$ given by (58), the mapping S is a contraction on X_T and bounded on Y_T. Assuming this for the moment, define $u_{-1} \equiv 0$ and $u_m = S(u_{m-1})$. If S is a contraction on X_T, we have $u_m \to u$ for some $u \in X_T$ which is therefore a weak solution of (1). If S is bounded on Y_T, Fatou's lemma shows that u must also belong to Y_T. Sobolev embedding given by the hyperbolic version of Lemma 4.2 implies $u \in L^\infty([\rho_0, T] \times H^n)$. Using standard local arguments one easily concludes $u \in C([\rho_0, T], \Lambda_s^2(H^n))$.

We prove first $S : Y_T \to Y_T$ is bounded provided $T \lesssim T_0$. The hyperbolic energy estimate, Lemma 4.4, and the definition of Y_T yield

$$\begin{aligned}\rho^{\frac{n}{2}} \|S(v)\|_{\Lambda_s^2(H^n)}(\rho) &\lesssim \epsilon + \int_{\rho_0}^{\rho} \sigma^{\frac{n}{2}} \|F(v)\|_{\Lambda_s^2(H^n)} d\sigma\\ &\lesssim \epsilon + \int_{\rho_0}^{\rho} \sigma^{\frac{n}{2}} \|v\|^p_{\Lambda_s^2(H^n)}(\sigma) d\sigma\\ &\lesssim \epsilon + \int_{\rho_0}^{\rho} \sigma^{\frac{n}{2}} \sigma^{-\frac{n}{2}p} \epsilon^p \, d\sigma.\end{aligned}$$

In order to conclude $\|S(v)\|_{Y_T} \lesssim \|v\|_{Y_T}$, we need only have

$$\begin{aligned}\int_{\rho_0}^{\rho} \sigma^{\frac{n}{2}} \sigma^{-\frac{n}{2}p} \epsilon^p d\sigma &\lesssim \epsilon\\ &\Rightarrow\ \rho \lesssim T_0.\end{aligned}$$

A completely analogous argument gives that $S : X_T \to X_T$ is a contraction in the X-norm:

$$\begin{aligned}\rho^{\frac{n}{2}} \|S(v_1) - S(v_2)\|_{L^2(H^n)}(\rho) &\lesssim \int_{\rho_0}^{\rho} \sigma^{\frac{n}{2}} \|F(v_1) - F(v_2)\|_{L^2(H^n)}(\sigma) d\sigma\\ &\lesssim \int_{\rho_0}^{\rho} \sigma^{\frac{n}{2}} \|(|v_1|^{p-1} + |v_2|^{p-1})|v_1 - v_2|\|_{L^2(H^n)} d\sigma\\ &\lesssim \|v_1 - v_2\|_{Y_T} \cdot \int_{\rho_0}^{\rho} \sigma^{-\frac{n}{2}(p-1)} \epsilon^{p-1} d\sigma.\end{aligned}$$

We can conclude that S is a contraction if

$$\rho^{-\frac{n}{2}p + \frac{n}{2} + 1} \lesssim \epsilon^{1-p}.$$

That is, $T \lesssim T_0$. We have shown

Proposition 9.2. *Consider the problem* (1) *where* $F = F(u)$ *satisfies* (2) *with* $p < 1 + \frac{2}{n}$.

With T_0 *given by* (58), *the lifespan* T_* *satisfies*

$$T_0 \quad \lesssim \quad T_* \quad \lesssim \quad T_0^{1+\gamma}$$

where $\gamma > 0$ *can be made arbitrarily small.*

10. Blowup for a semilinear equation with derivatives

We now consider non-linear Klein-Gordon equations in which the non-linearity F is allowed to depend on the time derivative u_t as well as u. In other words, we consider the problem

$$\left\{ \begin{array}{ll} (\Box - 1)u = F(u, u_t), & t > 2 \\ u(2, \cdot) = \varepsilon f, \ u_t(2, \cdot) = \varepsilon g & \end{array} \right. \tag{72}$$

for smooth data f, g supported in the unit ball, where F satisfies (2) in both variables. For simplicity we restrict ourselves to solutions with finite energy; that is, we assume that

$$u \in C(H^1) \cap C^1(L^2).$$

We will also assume finite speed of propogation, so that u is supported in the forward light cone Γ_+.

Theorem 1.1: *If* $n \geq 0$ *and* $1 < p \leq 1 + \frac{2}{n}$, *then there exists a scalar equation* (72) *with the non-linearity* F *satisfying* (2), *which does not have global finite-energy solution in* Γ_+ *for any non-zero smooth compactly supported data* (f, g) *and* $\varepsilon > 0$.

Proof Before we begin the actual proof, we consider the 0-dimensional scalar equation as motivation:

$$u_{tt} + u = -F(u, u_t), \quad u(2), u_t(2) = O(\varepsilon).$$

If we define the energy $E(t)$ by

$$E(t) = \frac{1}{2}|u_t|^2 + \frac{1}{2}|u|^2,$$

then we have

$$E'(t) = -F(u, u_t)u_t. \tag{73}$$

To create blow-up we must make the energy increase, and so it is plausible to choose the forcing term $-F$ to have the same sign as u_t. An obvious candidate for the blowup equation is therefore

$$u_{tt} + u = |u_t|^{p-1}u_t. \tag{74}$$

One may think of this equation as representing a child on a swing, always pushing to increase the energy of the system.

An informal argument that this equation blows up can be given as follows. ¿From (73) we have

$$E'(t) = |u_t|^{p+1}.$$

However, in the free harmonic oscillator the energy $E(t)$ is (on the average) evenly distributed between $|u_t|^2$ and $|u|^2$. Assuming that this equipartition also holds in the non-linear oscillator, we thus expect

$$E(t) \approx |u_t|^2 \tag{75}$$

in some averaged sense. Combining these two equations we see that

$$E'(t) \approx E(t)^{\frac{p+1}{2}}, \tag{76}$$

which should blow up in finite time as $\frac{p+1}{2} > 1$.

This argument works well when the non-linearity F is small. However, when u_t gets large the non-linearity begins to dominate the rest of the equation, and (74) ceases to behave like a harmonic oscillator. Indeed, for large u, u_t the equation (74) is an (unstable) perturbation of the equation $u = |u_t|^{p-1}u_t$, which admits global solutions with polynomial growth. Thus one expects global (but highly unstable) solutions to (74) for special choices of initial data, which frustrates any attempt to prove blowup from arbitrary data. (As an example of this phenomenon, we note that the equation

$$u_t t + u = |u_t|^2 u_t + \frac{3}{u_t},$$

which behaves qualitatively like (74) when u, u_t is large, has exact solutions $u(t) = \sqrt{\frac{8}{27}}(t+C)^{\frac{3}{2}}$ for all $C, t > 0$, even though the equation blows up in finite time for most choices of initial data.)

In the above situation we did not obtain blowup because (75) broke down, and the forcing term expended its effort fighting against the spring term u. In higher dimensions the problem is aggravated by the surface tension term $-\Delta u$, which also can fight against the forcing term. Thus, to make the previous blow-up argument work, we should try to preserve the equipartition relationship (75) even for large u, u_t. In other words, we need to keep the system behaving like a harmonic oscillator. To do this we shall make the spring term u so large as to dominate the forcing term $|u_t|^{p-1}u_t$. This suggests an equation of the form

$$u_{tt} + |u|^{p-1}u = |u_t|^{q-1}u_t \tag{77}$$

for some arbitrary $1 < q < p$. Depending on the relative strengths of u, u_t, and u_{tt} this equation should oscillate between

$$u_{tt} = |u_t|^{q-1}u_t$$

or

$$|u|^{p-1}u = |u_t|^{q-1}u_t,$$

and from the assumption $1 < q < p$, both of these equations will blow up in finite time.

¿From these (very informal) heuristics we are led to an equation which behaves like (74) for small u,u_t but behaves like (77) for large u, u_t. Accordingly, fix $1 < q < p$ and let $F(v)$ be a function obeying (2) such that $F(v)$ has the same sign as v, and

$$F(v) \sim \begin{cases} |v|^{p-1}v & |v| \lesssim 1 \\ |v|^{q-1}v & |v| \gtrsim 1. \end{cases} \tag{78}$$

We will show that the equation

$$\begin{aligned} (\Box - 1)u &= |u|^{p-1}u - F(u_t) \quad t > 2 \\ u(2,\cdot) &= \varepsilon f,\ u_t(2,\cdot) = \varepsilon g \end{aligned} \tag{79}$$

cannot have global H^1 solutions.

We will assume to the contrary that a global H^1 solution u does exist, and obtain a contradiction. As before, we define an energy functional $E(t)$ by

$$E(t) = \int_{\mathbf{R}^n} \frac{1}{2}|u_t|^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}|u|^2 + \frac{1}{p+1}|u|^{p+1}\ dx. \tag{80}$$

Since $p \leq 1 + \frac{2}{n} < 1 + \frac{4}{n-2}$, the problem is sub-critical with respect to H^1, and the above expression exists by Sobolev embedding.

A standard calculation using integration by parts gives

$$E'(t) = \int_{\mathbf{R}^n} F(u_t)u_t\ dx > 0. \tag{81}$$

In particular, this implies that $\int_{\mathbf{R}^n} F(u_t)u_t\ dx$ is locally integrable, and that the energy E is increasing and bounded away from zero.

To obtain substantial energy growth we need to somehow bound the right-hand side of (81) from below, preferably by some strongly increasing function of the energy $E(t)$. To do this we will need some analogue of the equipartition relationship (75). To this end we introduce the action $I(t)$ defined by

$$I(t) = \int_{\mathbf{R}^n} uu_t\ dx. \tag{82}$$

A standard calculation gives the equipartition identity

$$I'(t) = \int_{\mathbf{R}^n} |u_t|^2 - |\nabla u|^2 - |u|^2 - |u|^{p+1} + uF(u_t)\ dx.$$

Combining this with (80) we obtain a version of (75):

$$E(t) = \int_{\mathbf{R}^n} |u_t|^2 + \frac{1}{2}uF(u_t) - \frac{p-1}{2(p+1)}|u|^{p+1}\ dx - \frac{1}{2}I'(t).$$

We estimate the second and third terms using the elementary pointwise inequality

$$\frac{1}{2}uF(u_t) - \frac{p-1}{2(p+1)}|u|^{p+1} \leq C|F(u_t)|^{\frac{p+1}{p}}.$$

To deal with the fourth term we need to average in time. Integrating the above identity on $[T, T+A]$ for some $T, A > 0$ we get

$$\int_T^{T+A} E(t)\ dt \leq \int_T^{T+A} \int |u_t|^2 + C|F(u_t)|^{\frac{p+1}{p}}\ dxdt - \frac{1}{2}(I(T+A) - I(T)).$$

Using the definition of F in (78), the assumption $1 < q < p$, and the monotonicity of $E(t)$ we thus obtain

Proposition 10.1. *There exists $\varepsilon > 0$ such that,*

(83)

$$AE(T) \lesssim \int_T^{T+A} \int_{|u_t|\lesssim 1} |u_t|^2\ dxdt + \int_T^{T+A} \int_{|u_t|\gtrsim 1} |u_t|^{q+1-\varepsilon}\ dxdt + |I(T)| + |I(T+A)|$$

for all $T, A > 0$.

On the other hand, integrating (81) on $[T, T+A]$ and using (78) we obtain

(84)

$$\int_T^{T+A} \int_{|u_t|\lesssim 1} |u_t|^{p+1}\ dxdt + \int_T^{T+A} \int_{|u_t|\gtrsim 1} |u_t|^{q+1}\ dxdt \lesssim E(T+A) - E(T).$$

We will use (83) and (84), together with some easy estimates on $I(T)$, $I(T+A)$, to obtain blowup of the energy $E(T)$ in finite time, thus contradicting the existence of a global finite energy solution.

The first step will be to show that $E(t)$ exhibits at least polynomial growth. More precisely, we will prove that

$$\limsup_{T\to\infty} \frac{E(T)}{T^n} = +\infty. \tag{85}$$

Suppose this were not true, so that $E(T) \lesssim T^n$ for all T. Let A be a large constant to be chosen later, and suppose that $T \gg A$. From (84) and Hölder's inequality we have

$$\int_T^{T+A} \int_{|u_t|\lesssim 1} |u_t|^2\ dxdt \lesssim (E(T+A) - E(T))^{\frac{2}{p+1}} (AT^n)^{1-\frac{2}{p+1}}.$$

Also, from (84) we trivially have

$$\int_T^{T+A} \int_{|u_t|\gtrsim 1} |u_t|^{q+1-\varepsilon}\ dxdt \lesssim E(T+A) - E(T).$$

Finally, from (82) the Cauchy-Schwarz inequality and (80) we have

$$|I(t)| \leq (\int_{\mathbf{R}^n} |u|^2 \, dx)^{1/2} (\int_{\mathbf{R}^n} |u_t|^2 \, dx)^{1/2} \lesssim E(t).$$

Inserting these three estimates into (83) and using monotonicity of the energy we obtain

$$AE(T) \lesssim (E(T+A)-E(T))^{\frac{2}{p+1}} (AT^n)^{1-\frac{2}{p+1}} + (E(T+A)-E(T)) + E(T) + E(T+A).$$

Subtracting a constant multiple of $E(T)$ from both sides gives (with A a sufficiently large constant)

$$E(T) \lesssim (E(T+A)-E(T))^{\frac{2}{p+1}} (T^n)^{1-\frac{2}{p+1}} + (E(T+A)-E(T)).$$

Since we are assuming $E(T) \lesssim T^n$, we have $E(T+A) - E(T) \lesssim T^n$, and so the first term on the right is larger than the second. Thus we are reduced to

$$E(T) \lesssim (E(T+A)-E(T))^{\frac{2}{p+1}} T^{\frac{n(p-1)}{p+1}},$$

which can be rearranged as

$$E(T+A) - E(T) \gtrsim E(T)^{\frac{p+1}{2}} T^{-\frac{n(p-1)}{2}}. \tag{86}$$

This is the analogue of (76). We claim that (86) contradicts the assumption $E(T) \lesssim T^n$. To see this, we first observe that (86) can be improved to

$$E(T+A) - E(T) \gtrsim E(T+A)^{\frac{p+1}{2}} T^{-\frac{n(p-1)}{2}}. \tag{87}$$

If $E(T+A) \leq 2E(T)$ this follows immediately from (86). However, if $E(T+A) > 2E(T)$, then

$$E(T+A) - E(T) \geq \frac{1}{2} E(T+A) \gtrsim E(T+A)^{\frac{p+1}{2}} T^{-\frac{n(p-1)}{2}},$$

where the last inequality follows from the assumption $E(T+A) \lesssim T^n$.

¿From the mean-value theorem applied to $f(x) = x^{\frac{1-p}{2}}$ and energy monotonicity we see that

$$E(T)^{\frac{1-p}{2}} - E(T+A)^{\frac{1-p}{2}} \geq \frac{p-1}{2} (E(T+A) - E(T)) E(T+A)^{-\frac{p+1}{2}}.$$

Combining this with (87) gives

$$E(T)^{\frac{1-p}{2}} - E(T+A)^{\frac{1-p}{2}} \gtrsim T^{-\frac{n(p-1)}{2}}. \tag{88}$$

The left-hand side is a telescopic series and therefore summable. However the right-hand side is divergent since $p \leq 1 + \frac{2}{n}$. This is a contradiction, and so (85) must hold.

By (85) we can find a large time T_0 such that $E(T_0) \geq BT_0^n$, where B is a large constant to be chosen later. We will show that $E(t)$ blows up before time $2T_0$, contradicting the assumption of finite energy.

To do this we return to (83), but estimate the various terms in a different manner from before. Let $T_0 \leq T \leq 2T_0$, and suppose that $0 < A \ll 1$ is small. We have the trivial estimate

$$\int_T^{T+A} \int_{|u_t| \lesssim 1} |u_t|^2 \, dxdt \lesssim AT^n,$$

and from (84) and Hölder's inequality we have

$$\int_T^{T+A} \int_{|u_t| \gtrsim 1} |u_t|^{q+1-\varepsilon} \, dxdt \lesssim (E(T+A) - E(T))^{1-\varepsilon_1} (AT^n)^{\varepsilon_1}$$

for some $\varepsilon_1 > 0$. Finally, from (82) and Hölder's inequality we have

$$|I(t)| \lesssim (t^n)^{\frac{1}{2} - \frac{1}{p+1}} \left(\int_{\mathbf{R}^n} |u|^{p+1} \, dx\right)^{\frac{1}{p+1}} \left(\int_{\mathbf{R}^n} |u_t|^2 \, dx\right)^{\frac{1}{2}},$$

so by (80) we have

$$|I(t)| \lesssim E(t)^{1-\varepsilon_2} (t^n)^{\varepsilon_2}$$

for some $\varepsilon_2 > 0$. Inserting these estimates into (83) as before, we obtain

$$(89)\, AE(T) \lesssim AT^n + (E(T+A) - E(T))^{1-\varepsilon_1} (AT^n)^{\varepsilon_1} + E(T+A)^{1-\varepsilon_2} (T^n)^{\varepsilon_2}.$$

Fix T, and suppose that A is shorter than the doubling time of the energy $E(T)$, i.e. $E(T+A) < 2E(T)$. Then by dividing (89) by $AE(T)$ we obtain

$$1 \lesssim \frac{T^n}{E(T)} + \frac{T^{n\varepsilon_1}}{A^{1-\varepsilon_1} E(T)^{\varepsilon_1}} + \frac{T^{n\varepsilon_2}}{AE(T)^{\varepsilon_2}}.$$

Since $E(T) \gtrsim BT^n$, we thus obtain (if B is sufficiently large)

$$A \lesssim T_0^{n\varepsilon} E(T)^{-\varepsilon}$$

for some $\varepsilon > 0$. In other words,

$$(90) \qquad E(T+A) \geq 2E(T) \text{ whenever } T_0 \leq T \leq 2T_0 \text{ and } a \gtrsim T_0^{n\varepsilon} E(T)^{-\varepsilon}.$$

Write $E(T_0) = E_0$. From (90) and induction we see that

$$E(T_0 + A) > 2^k E_0 \text{ whenever } T_0 \geq A \gtrsim \sum_{j=0}^{k-1} T_0^{n\varepsilon} (2^k E_0)^{-\varepsilon j}.$$

However, since $E_0 \geq BT_0^n$, the summation is $O(1)$ for all k. Since T_0 was assumed to be large, we see that $E(2T_0) > 2^k E_0$ for all k, contradicting the finite energy assumption. This finishes the proof. ■

One can go through the above argument carefully and obtain a quantitative upper bound on the time of existence; we sketch the argument as follows. It is enough to obtain upper bounds on T_0, the first time T for which $E(T) \geq BT^n$. To do this we use the fact that (88) holds for $T < T_0$.

The left-hand side of (88) is telescopic and sums to $O(\varepsilon^{1-p})$. The right-hand side sums like $T_0^{1-n\frac{p-1}{2}}$ if $p < 1+\frac{2}{n}$, or $\log T_0$ if $p = 1+\frac{2}{n}$. Thus one has

$$T_0 \lesssim \varepsilon^{-\frac{2(p-1)}{2-np+n}} \tag{91}$$

if $p < 1+\frac{2}{n}$, and

$$T_0 \lesssim e^{C\varepsilon^{-(p-1)}}. \tag{92}$$

if $p = 1+\frac{2}{n}$.

We expect the bounds in (91), (92) to be in general sharp. This sharpness is in some cases known. For the example $p = 3, n = 1$, [14] proves a lower bound $T_0 \gtrsim e^{C\varepsilon^{-2}}$ and conjectures the upper bound (92).

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