The cosmic distance ladder

Teacher's day, AustMS06

27 September 2006

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Astrometry

An important subfield of astronomy is astrometry, the study of positions and movements of celestial bodies (sun, moon, planets, stars, etc.). Typical questions in astrometry are:

- How far is it from the Earth to the Moon?
- From the Earth to the Sun?
- From the Sun to other planets?
- From the Sun to nearby stars?
- From the Sun to distant stars?

- Of course, these distances are far too vast to be measured directly.
- Nevertheless we have many indirect ways of computing these distances.
- These methods are often very clever, relying not on technology but rather on observation and high-school mathematics.
- Usually, the indirect methods control large distances in terms of smaller distances. One then needs more methods to control these distances in terms of even smaller distances, until one gets down to distances that one can measure directly. This is the cosmic distance ladder.

First rung: the radius of the earth

- Nowadays, we know that the earth is approximately spherical, with radius 6378 kilometres at the equator and 6356 kilometres at the poles. These values have now been verified to great precision by many means, including modern satellites.
- But suppose we had no advanced technology such as spaceflight, ocean and air travel, or even telescopes and sextants. Would it still be possible to convincingly argue that the earth must be (approximately) a sphere, and to compute its radius?

The answer is yes - if one knows geometry!

- Aristotle (384–322 BCE) gave a simple argument demonstrating why the Earth is a sphere (which was first asserted by Parmenides (515–450 BCE)).
- Eratosthenes (276–194 BCE) computed the radius of the Earth at $40,000$ stadia (about 6800 kilometres). As the true radius of the earth is 6356–6378 kilometres, this is only off by eight percent!

Aristotle's argument

- Aristotle reasoned that lunar eclipses were caused by the Earth's shadow falling on the moon. This was because at the time of a lunar eclipse, the sun was always diametrically opposite the earth (this could be measured by timing the sun's motion, or by using the constellations ("fixed stars") as reference).
- Aristotle also observed that the terminator (boundary) of this shadow on the moon was always a circular arc, no matter what the positions of the Earth, Moon, and Sun were. Thus every projection of the Earth was a circle, which meant that the Earth was most likely a sphere. For instance, Earth could not be a disk, because the shadows would usually be elliptical arcs rather than circular ones.

Eratosthenes' argument

- Aristotle also argued that the Earth's radius could not be incredibly large, because it was known that some stars could be seen in Egypt but not in Greece, or vice versa. But this did not give a very accurate estimate on the Earth's radius.
- Eratosthenes gave a more precise argument. He had read of a well in Syene, which lay to the south of his home in Alexandria, of a deep well which at noon on the summer solstice (June 21) would reflect the sun overhead. (This is because Syene happens to lie almost exactly on the tropic of Cancer.)
- Eratosthenes then observed a well in Alexandria at June 21, but found that the sun did not reflect off the well at noon; using a gnomon (a measuring stick) and some elementary trigonometry, he found instead that the sun was at an angle of about $7°$ from the vertical.

- Information from trade caravans and other sources established the distance between Alexandria and Syene to be about 5000 stadia (about 740 kilometres). This is the only direct measurement used here, and can be thought of as the "zeroth rung" on the cosmic distance ladder.
- Eratosthenes also assumed the sun was very far away compared to the radius of the earth (more on this in the "third rung" section).
- High school trigonometry then suffices to establish an estimate for the radius of the earth.

Second rung: shape, size, and location of the moon

- What is the shape of the moon?
- What is the radius of the moon?
- How far is the moon from the earth?

Again, these questions were answered with remarkable accuracy by the ancient Greeks.

- Aristotle argued that the moon was a sphere (rather than a disk) because the terminator (the boundary of the sun's light on the moon) was always a circular arc.
- Aristarchus (310–230 BCE) computed the distance of the Earth to the Moon as about 60 Earth radii. (Indeed, the distance varies between 57 and 63 Earth radii due to eccentricity of the orbit.)
- Aristarchus also estimated the radius of the moon as one third the radius of the earth. (The true radius is 0.273 Earth radii.)
- The radius of the earth is of course known from the preceding rung of the ladder.

How did Aristarchus do it?

- Aristarchus knew that lunar eclipses were caused by the shadow of the Earth, which would be roughly two Earth radii in diameter. (This assumes the sun is very far away from the earth; more on this in the "third rung" section.)
- From many observations it was known that lunar eclipses last a maximum of three hours.
- It was also known that the moon takes one month to make a full rotation of the earth.
- From this and basic algebra Aristarchus concluded that the distance of the Earth to the Moon was about 60 Earth radii.

- The moon takes about a 2 minutes (1/720 of a day) to set. Thus the angular width of the moon is 1/720 of a full angle, or about $\frac{1}{2}^\circ$.
- Since Aristarchus knew the moon was 60 Earth radii away, basic trigonometry then gives the radius of the moon as about 1/3 Earth radii. (Aristarchus was handicapped, among other things, by not possessing an accurate value for π, which had to wait until Archimedes (287–212 BCE) some decades later!)

Third rung: size and location of the sun

- What is the radius of the sun?
- How far is the the sun from the earth?

Once again, the ancient Greeks could answer this question!

- Aristarchus already knew that the radius of the moon was about 1/180 of the distance to the moon. Since the sun and moon have about the same angular width (most dramatically seen during a solar eclipse), he concluded that the radius of the sun is 1/180 of the distance to the sun. (The true answer is 1/215.)
- Aristarchus estimated the sun as roughly 20 times further than the moon. This turned out to be inaccurate (the true factor is roughly 390), because the mathematical method, while technically correct, was very unstable. Hipparchus (190–120 BCE) and Ptolemy (90–168 CE) obtained the slightly more accurate ratio of 42.
- Nevertheless, these results were enough to establish that the important fact that the Sun was much larger than the Earth.

Because of this Aristarchus proposed the heliocentric model more than 1700 years before Copernicus! (Copernicus credits Aristarchus for this in his own, more famous work.)

- Ironically, Aristarchus's heliocentric model was dismissed by later Greek thinkers, for reasons related to the sixth rung of the ladder (see below).
- Since the distance to the moon was established on the preceding rung of the ladder, we now know the size and distance to the sun. (The latter is known as the Astronomical Unit (AU), and will be fundamental for the next three rungs of the ladder).

How did this work?

- Aristarchus knew that each new moon was one lunar month after the previous one.
- By careful observation, Aristarchus also knew that a half-moon occured slightly earlier than the midpoint between a new moon and full moon; he measured this discrepancy as 12 hours. (Alas, it is difficult to measure a half-moon perfectly, and the true discrepancy is 1/2 an hour.)
- Elementary trigonometry then gives the distance to the sun as roughly 20 times the distance to the moon.

Fourth rung: distances from the sun to the planets

Now we consider other planets, such as Mars. The ancient astrologers already knew that the sun and planets stayed within the Zodiac, which implied that the solar system essentially lay on a two-dimensional plane (the ecliptic). But there are many further questions:

- How long does Mars take to orbit the sun?
- What shape is the orbit?
- How far is Mars from the sun?

- These questions were attempted by Ptolemy, but with extremely inaccurate answers (in part due to the use of the Ptolemaic model of the solar system rather than the heliocentric one).
- Copernicus (1473–1543) estimated the (sidereal) period of Mars as 687 days and its distance to the sun as 1.5 AU. Both measures are accurate to two decimal places. (Ptolemy obtained 15 years (!) and 4.1 AU.)
- It required the accurate astronomical observations of Tycho Brahe (1546–1601) and the mathematical genius of Johannes Kepler (1571–1630) to find that Earth and Mars did not in fact orbit in perfect circles, but in ellipses. This and further data led to Kepler's laws of motion, which in turn inspired Newton's theory of gravity.

How did Copernicus do it?

- The Babylonians already knew that the apparent motion of Mars repeated itself every 780 days (the synodic period of Mars).
- The Copernican model asserts that the earth revolves around the sun once every solar year (365 days).
- Subtracting the two implied angular velocities yields the true (sidereal) Martian period of 687 days.
- The angle between the sun and Mars from the Earth can be computed using the stars as reference. Using several measurements of this angle at different dates, together with the above angular velocities, and basic trigonometry, Copernicus computed the distance of the Mars to the sun as approximately 1.5 AU.

Kepler's problem

- Copernicus's argument assumed that Earth and Mars moved in perfect circles. Kepler suspected this was not the case - it did not quite fit Brahe's observations - but how to then find the correct orbit of Mars?
- Brahe's observations gave the angle between the sun and Mars from Earth very accurately. But the Earth is not stationary, and might not move in a perfect circle. Also, the distance from Earth to Mars remained unknown. Computing the orbit of Mars precisely from this data seems hopeless - not enough information!

To solve this problem, Kepler came up with two extremely clever ideas.

- **To compute the orbit of Mars accurately, first compute the orbit of Earth accurately**. If you know exactly where the Earth is at any given time, the fact that the Earth is moving can be compensated for by mathematical calculation.
- **To compute the orbit of Earth, use Mars itself as a fixed point of reference!** To pin down the location of the Earth at any given moment, one needs two measurements (because the plane of the solar system is two dimensional). The direction of the sun (against the stars) is one measurement; the direction of Mars is another. But Mars moves!

- Kepler's breakthrough was to take measurements spaced 687 days apart, when Mars returns to its original location and thus serves as a fixed point. Then one can triangulate between the Sun and Mars to locate the Earth. Once the Earth's orbit is computed, one can invert this trick to then compute Mars' orbit also.
- Albert Einstein (1879–1955) referred this idea of Kepler's as "an idea of pure genius".
- Similar ideas work for the other planets. Since the AU is already deducible from previous rungs of the ladder, we now have distances to all the planets.
- Around 1900, when travel across the Earth became relatively easy, parallax methods could compute these distances more directly and accurately, confirming and strengthening all the rungs so far of the distance ladder.

Fifth rung: the speed of light

- Technically, the speed of light is not a distance. However, one of the first accurate measurements of this speed came from the fourth rung of the ladder, and knowing the value of this speed is important for later rungs.
- Ole Rømer (1644–1710) and Christiaan Huygens (1629–1695) obtained a value of $220,000$ km/sec, close to but somewhat less than the modern value of $299,792$ km/sec, using Io's orbit around Jupiter.

How did they do it?

- Rømer observed that Io rotated around Jupiter every 42.5 hours, by timing when Io entered and exited Jupiter's shadow.
- But the period was not uniform; when the Earth moved from being aligned with Jupiter to being opposed to Jupiter, the period had lagged by about 20 minutes. He concluded that light takes 20 minutes to travel 2 AU. (It actually takes about 17 minutes.)
- Huygens combined this with a precise (for its time) computation of the AU to obtain the speed of light.
- Nowadays, the most accurate measurements of distances to planets are obtained by radar, which requires precise values of the speed of light. This speed can now be computed very accurately by terrestrial means, thus giving more external support to the distance ladder.

The data collected from these rungs of the ladder have also been decisive in the further development of physics and in ascending higher rungs of the ladder.

- The accurate value of the speed of light (as well as those of the permittivity and permeability of space) was crucial in leading James Clerk Maxwell to realise that light was a form of electromagnetic radiation. From this and Maxwell's equations, this implied that the speed of light in vacuum was a universal constant c in every reference frame for which Maxwell's equations held.
- Albert Einstein reasoned that Maxwell's equations, being a fundamental law of physics, should hold in every inertial reference frame. The above two hypotheses lead inevitably to the special theory of relativity. This theory becomes important in the ninth rung of the ladder (see below) in order to relate red shifts with velocities accurately.

- Accurate measurements of the orbit of Mercury revealed a slight precession in its elliptical orbit, suggesting that a refinement was needed to the theories of Kepler and Newton. This provided one of the very first experimental confirmations of Einstein's general theory of relativity. This theory is also crucial at the ninth rung of the ladder.
- Maxwell's theory that light is a form of electromagnetic radiation also helped develop the important astronomical tool of spectroscopy, which becomes important in the seventh and ninth rungs of the ladder (see below).

Sixth rung: distances to nearby stars

- By taking measurements of the same star six months apart and comparing the angular deviation, one obtains the distance to that star as a multiple of the Astronomical Unit. This parallax idea, which requires fairly accurate telescopy, was first carried out successfully by Friedrich Bessel (1784–1846) in 1838.
- It is accurate up to distances of about 100 light years (≈ 30 parsecs). This is enough to locate several thousand nearby stars.
- Ironically, the ancient Greeks dismissed Aristarchus's estimate of the AU and the heliocentric model that it suggested, because it would have implied via parallax that the stars were an inconceivably enormous distance away. (Well... they are.)

Seventh rung: distances to moderately distant stars

- Twentieth-century telescopy could easily compute the apparent brightness of stars. Combined with the distances to nearby stars from the previous ladder and the inverse square law, one could then infer the absolute brightness of nearby stars.
- Ejnar Hertzsprung (1873–1967) and Henry Russell (1877–1957) plotted this absolute brightness against colour in 1905–1915, leading to the famous Hertzsprung-Russell diagram relating the two. Now one could measure the colour of distant stars, hence infer absolute brightness; since apparent brightness could also be measured, one can solve for distance.
- This method works up to $300,000$ light years! Beyond that, the stars in the HR diagram are too faint to be measured accurately.

Eighth rung: distances to very distant stars

- Henrietta Swan Leavitt (1868–1921) observed a certain class of stars (the Cepheids) oscillated in brightness periodically; plotting the absolute brightness against the periodicity she observed a precise relationship. This gave yet another way to obtain absolute brightness, and hence observed distances.
- Because Cepheids are so bright, this method works up to $13,000,000$ light years! Most galaxies are fortunate enough to have at least one Cepheid in them, so we know the distances to all galaxies out to a reasonably large distance.
- Beyond that scale, only ad hoc methods of measuring distances are known (e.g. relying on supernovae measurements, which are one of the few events that can still be detected at such distances).

Ninth rung: the shape of the universe

- Combining all the above data against more precise red-shift measurements, together with the known speed of light (see fifth rung) Edwin Hubble (1889–1953) formulated the famous Hubble's law relating velocity (as observed by redshift) with distance, which led in turn to the famous "Big Bang" model of the expanding universe. This law can be then used to give another measurement of distance at the largest scales (though one which is subject to a number of other distorting effects).
- These measurements have led to accurate maps of the universe at large scales, which have led in turn to many discoveries of very large-scale structures which would not have been possible without such good astrometry (the Great Wall, Great Attractor, etc.) For instance, our best estimate of the current

diameter of the observable universe is now about 78 billion light-years.

- The mathematics becomes more advanced at this point, as the effects of general relativity become very important. Conversely, the development of general relativity has been highly influenced by the data we have at this scale of the universe. Cutting-edge technology (such as the Hubble space telescope) has also been vital to this effort.
- Climbing this rung of the ladder (i.e. mapping the universe at its very largest scales) is still a very active area in astronomy today!

Acknowledgements

- Thanks to Richard Brent for corrections and comments.
- Much of the data here was collected from various internet sources (usually starting with Wikipedia and then branching out to more primary source material).