A bilinear approach to the Restriction and Kakeya conjectures

Terence Tao

Mathematics Department

UCLA

With Ana Vargas and Luis Vega

`www.math.ucla.edu/~tao/preprints`

The purpose of this talk is to:

- Describe the linear restriction conjecture, and its bilinear variant
- Discuss the equivalences between the linear and bilinear versions, and the extra estimates available in the bilinear setting
- Show how the bilinear perspective on the restriction (and Kakeya) conjectures can be used to yield new progress on the linear restriction conjecture

Let $n \geq 2$, and let S be a smooth compact hypersurface in $\mathbf{R}^n$ with surface measure $d\sigma$. Suppose that f is an L^p measure on S. We ask whether the Fourier transform $\widehat{fd\sigma}$ of f is necessarily in an L^q class. More precisely, we ask whether one has the a priori estimate

$$\|\widehat{fd\sigma}\|_{L^q(\mathbf{R}^n)} \lesssim \|f\|_{L^p(S)}$$

for test functions f. If this is the case we say that $R^*(p \to q)$ holds.

- This is trivial if $q = \infty$. Also, if it is true for some (p, q) it is also true for higher p and higher q by Sobolev and Hölder inequalities.
- If S is flat then no further estimates are possible, because $\widehat{fd\sigma}$ will not decay in the normal direction. However, for curved surfaces more estimates exist.
- These estimates are important to PDE: if S is the light cone (or sphere, or parabola, or cubic), then $\widehat{fd\sigma}$ is a solution to the wave equation (or Helmholtz, or Schrödinger equation, or KdV).
- For simplicity we restrict ourselves to the paraboloid $x_n = \frac{1}{2}|\underline{x}|^2$, $|x| \lesssim 1$ in $\mathbf{R}^n$ for $n = 2, 3$, and ignore epsilon losses in the p, q indices.

Necessary conditions on p, q

- If one takes $f = 1$, then $\widehat{f d\sigma}$ decays like $|x|^{-(n-1)/2}$, and so one needs $q > 2n/(n-1)$.
- If one takes f to be a bump function on a cap (e.g. $|\underline{x}| \lesssim \delta$), then $\widehat{f d\sigma}$ is about δ^{n+1} on a $\delta^{-1} \times \delta^{-2}$ tube, and so one needs $q \geq (n+1)/(n-1)p'$. (The "Knapp example").
- For the paraboloid, this condition also can be obtained from scaling considerations (using the natural parabolic scaling associated to S).
- The *restriction conjecture* states that these two conditions are also sufficient for $R^*(p \to q)$.
- When $q = 2p'$ we say that the estimate $R^*(p \to q)$ is *sharp*. Sharp estimates have the advantage of being scale-invariant, and so (for instance) can be proven for the entire paraboloid and not just a compact subset.

The two-dimensional case

- When $n = 2$, the restriction conjecture has been proved. We sketch the argument of Fefferman (ignoring epsilons and logarithmic divergences for simplicity).
- It suffices to prove the estimate $R^*(4 \to 4)$, since the other estimates in the restriction conjecture can be obtained by interpolation and Hölder's inequality.
- We have to prove

$$\|\widehat{f d\sigma}\|_4 \lesssim \|f\|_4$$

when $d\sigma$ is surface measure on an arc of the parabola in $\mathbf{R}^2$.

- We square both sides and apply Plancherel's theorem:

$$\|f d\sigma * f d\sigma\|_2 \lesssim \|f\|_4 \|f\|_4.$$

- Now we use the geometric fact that every point $\xi \in \mathbf{R}^2$ is the sum of at most one pair of points ω_1, ω_2 in S. In fact, we can say that

$$f d\sigma * f d\sigma(\omega_1 + \omega_2) \sim f(\omega_1) f(\omega_2) / |\omega_1 - \omega_2|.$$

So after a change of variables we have

$$\|fd\sigma * fd\sigma\|_2^2 \sim \int\int |f(\omega_1)|^2 |f(\omega_2)|^2 \frac{d\omega_1 d\omega_2}{|\omega_1 - \omega_2|},$$

and estimate follows from fractional integration.

- We have just proved the estimate

$$\|\widehat{fd\sigma}\widehat{fd\sigma}\|_2 \lesssim \|f\|_4\|f\|_4.$$

This is obviously equivalent to the bilinear estimate

$$\|\widehat{fd\sigma}\widehat{gd\sigma}\|_2 \lesssim \|f\|_4\|g\|_4.$$

Without any further conditions on f and g, this estimate is sharp (just take $f = g =$ the Knapp example).

- However, if we make the restriction that f and g are supported on *disjoint* (unit-separated) portions on the parabola, then one can do better. Indeed, by repeating Fefferman's argument we have

$$\|\widehat{fd\sigma}\widehat{gd\sigma}\|_2^2 \lesssim \int\int |f(\omega_1)|^2|g(\omega_2)|^2 \frac{d\omega_1 d\omega_2}{|\omega_1 - \omega_2|}$$

and since $|\omega_1 - |\omega_2| \sim 1$ we thus have

$$\|\widehat{fd\sigma}\widehat{gd\sigma}\|_2 \lesssim \|f\|_2\|g\|_2$$

which is a superior estimate.

Definition: We let $R^*(p \times p \to q)$ denote the estimate

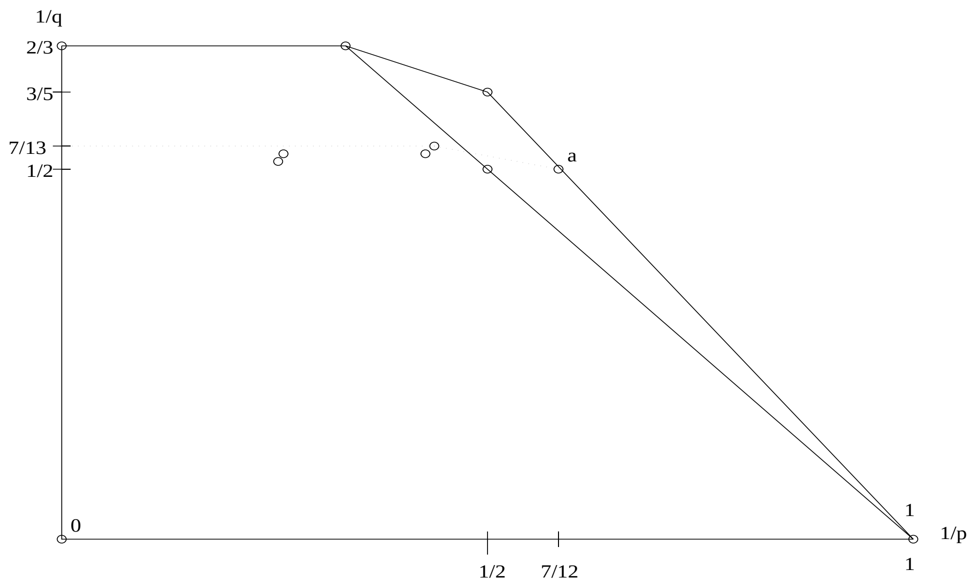
$$\|\widehat{fd\sigma}\widehat{gd\sigma}\|_q \lesssim \|f\|_p \|g\|_p.$$

whenever f, g lie on unit-separated subsets of S.

- By Hölder's inequality, $R^*(p \to q)$ implies $R^*(p \times p \to q/2)$, however the converse is not true because of the separation condition. On the previous slide we have proved $R^*(2 \times 2 \to 2)$, despite the fact that the corresponding linear estimate $R^*(2 \to 4)$ is false.
- Fortunately, a satisfactory converse can be proven: If $R^*(p \times p \to q/2 - \varepsilon)$ holds and $q \leq (n+1)/(n-1)p'$, then $R^*(p \to q)$ holds. Recall that $q \leq (n+1)/(n-1)p'$ was a necessary condition for $R^*(p \to q)$ (but not for $R^*(p \times p \to q/2)$).
- Thus the bilinear estimates include the linear estimates as a subset, but are not subject to the scaling condition (aka the Knapp example) and so encompass a richer set of estimates.

Bilinear restriction estimates, $n = 3$.

- In the bilinear setting, replacing $R^*(p \to q)$ by $R^*(p \times p \to q/2)$, we expect a wider range of estimates because the Knapp counterexample is much less of a nuisance. Indeed, a few simple examples ("squashed Knapp examples") lead one to expect the bilinear estimate to hold in a pentagonal region which is obtained by interpolating the restriction conjecture estimates with a bilinear estimate $R^*(2 \times 2 \to 5/3)$, which was first conjectured by Machedon and Klainerman.



Sketch of bilinear to linear implication

- Assume that we have the bilinear estimate $R^*(p \times p \to q/2 - \varepsilon)$

$$\|\widehat{fd\sigma}\widehat{gd\sigma}\|_{q/2-\varepsilon} \lesssim \|f\|_p \|g\|_p$$

for separated f, g, and we want to prove the estimate

$$\|\widehat{fd\sigma}\|_q \lesssim \|f\|_p$$

(ignoring epsilons). We square this as

$$\|\widehat{fd\sigma}\widehat{fd\sigma}\|_{q/2} \lesssim \|f\|_p \|f\|_p.$$

- Break up f into two pieces f_1, f_2 supported on different halves of S. Roughly speaking, the contribution of

$$\|\widehat{f_1 d\sigma}\widehat{f_2 d\sigma}\|_{q/2}$$

can be controlled by the bilinear hypothesis since f_1, f_2 are essentially separated. So we only need to deal with the diagonal terms

$$\|\widehat{f_1 d\sigma}\widehat{f_1 d\sigma}\|_{q/2}$$

$$\|\widehat{f_2 d\sigma}\widehat{f_2 d\sigma}\|_{q/2}$$

From Planchere's theorem these functions have disjoint Fourier supports and so we can exploit some orthogonality. To estimate each term, we take advantage of the fact that the paraboloid is invariant under parabolic scaling to rescale f_1 and f_2 to live on the original surface S (not just one half of S). Then we iterate the entire process, decomposing f_1, f_2, etc. Providing we have the scaling condition $q \leq (n+1)/(n-1)p'$, this process converges (modulo a technicality which requires the epsilon in the hypothesis).

Bilinear and linear restriction theorems in three dimensions

- In three dimensions the restriction conjecture is not completely solved. The ideal estimate would be $R^*(3 \to 3)$ (ignoring epsilons), but we are some way from achieving that goal. In the 1970s Tomas and Stein (with independent work by Strichartz and Sjölin) proved the estimate $R^*(2 \to 4)$, which is sharp with respect to the scaling condition. This estimate was not improved until the 1990s by Bourgain, Wolff, and Muyoa, Vargas, and Vega; for instance, the latter three authors proved $R^*(7/3 \to 4 - 2/11 + \varepsilon)$. These estimates improved the 4 exponent of Tomas-Stein, but worsened the 2 exponent; in particular the estimates were no longer sharp with respect to scaling.
- Our first application of the bilinear hypothesis is to show that there are improvements to Tomas-Stein which are sharp with respect to scaling.

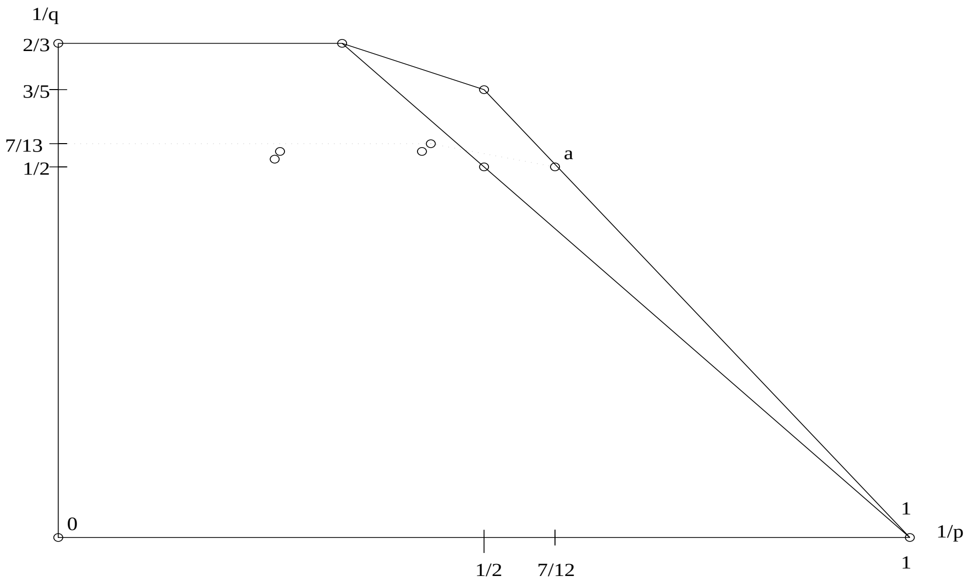
First step: a bilinear improvement to Tomas-Stein

- The Tomas-Stein estimate $R^*(2 \to 4)$ is sharp in the sense that the 2 exponent cannot be lowered without raising the 4 exponent, as can be seen from scaling/Knapp. However, the bilinear version $R^*(2\times 2 \to 2)$ of Tomas Stein is not sharp, and it is possible to improve the indices.
- Theorem (Muyoa, Vargas, Vega): $R^*(12/7 \times 12/7 \to 2)$.
- The proof begins like the Fefferman argument: to prove
$$\|\widehat{fd\sigma}\widehat{gd\sigma}\|_2 \lesssim \|f\|_{12/7}\|f\|_{12/7}$$
it suffices by Plancherel to prove
$$\|fd\sigma * gd\sigma\|_2 \lesssim \|f\|_{12/7}\|f\|_{12/7}$$
Because we are now in three dimensions instead of two, the convolution is a bit more complicated than before:
$$fd\sigma * gd\sigma(\xi) = \int_{\Gamma_\xi} f(\eta)g(\xi-\eta)\, d\eta$$
where Γ_ξ is some ellipse depending on ξ in a way which satisfies the cinematic curvature condition.

- The estimate then follows from the well-developed theory of L^p, L^q estimates for Radon transforms.

Second step: using the linear-bilinear equivalence

- By interpolating the bilinear improvement of Tomas-Stein with other improvements to Tomas-Stein, we can obtain new sharp restriction theorems.
- We convert the linear improvements to Tomas-Stein to bilinear improvements, then by interpolation we obtain bilinear improvements on the line of scaling. Then we apply the bilinear-linear equivalence.



More new restriction theorems

- In addition to obtaining new restriction theorems from interpolation, we have also proven new restriction theorems which are not interpolants.

- The restriction theorem $R^*(7/3 \to 2 - 2/11)$ obtained by the work of Bourgain, Wolff, Muyoa, Vargas, and Vega is based on a complex argument by Bourgain, which combines a variant of the Tomas-Stein argument with a version of Cordoba's proof of the Bochner-Riesz conjecture in two dimensions. It requires two estimates as input: a "local" restriction theorem, which was obtained in the above work by interpolating the Tomas-Stein estimate with an estimate coming from the Sobolev trace lemma, and a Kakeya estimate (of which the best known result in three dimensions is due to Wolff).

- By converting Bourgain's argument to a bilinear argument we can improve the local restriction theorem imput by replacing Tomas-Stein with the bilinear improvement to Tomas-Stein.

- One can also find bilinear improvement's to Wolff's Kakeya result by inspecting the Wolff's proof. Also there is an x-ray estimate of Wolff which is slightly stronger than the Kakeya result, and which can also be used as a substitute for Kakeya in this argument when combined with the bilinear improvements to Tomas Stein.

- Combining all these improvements together we have managed to improve the restriction theorem exponents from $R^*(7/3 \to 2 - 2/11 + \varepsilon)$ to $R^*(26/11 \to 2 - 2/7 + \varepsilon)$

Restriction theorems for the cone

- The above restriction theorems were stated for paraboloids, but also apply to any surface of nonzero curvature. The situation becomes more subtle when one of the principal curvatures is allowed to vanish.

- In the case of the cylinder there are no non-trivial linear restriction theorems, and the best possible bilinear restriction theorem is $R^*(2 \times 2 \to 2)$, which is fairly easy to prove.

- In the case of the cone, the best possible linear theorem is $R^*(4 \to 4)$, which is also fairly easy to prove. One can also show $R^*(2 \times 2 \to 2)$, but this is not sharp, in 1995 Bourgain improved this to $R^*(2 \times 2 \to 2 - 13/2408 + \varepsilon)$. Machedon and Klainerman have conjectured that $R^*(2 \times 2 \to 5/3)$; this is best possible.

- The problem for the cone is more difficult than that for the surfaces with non-zero curvature, partly because of the lack of Kakeya estimates, and partly because the Fourier transform of the cone decays more slowly than that of the paraboloid or similar surfaces.

- By recasting Bourgain's arguments using our techniques, we have managed to improve the cone restriction theorem to $R^*(2 \times 2 \to 2 - 8/121 + \varepsilon)$.

Further consequences

- The bilinear estimates proved above can be used to improve progress on several open problems in harmonic analysis and PDE. For instance:
- We can prove pointwise convergence of the solution of the free Schrödinger equation in R^{2+1} to its initial data if the data is in H^s for $s > 1/2 - 1/32$. This improves upon previous work of Muyoa, Vargas, and Vega.
- We can obtain some null form estimates for the wave equation for L^p in R^{2+1} for $2 > p > 2 - 8/121$, of the type studied by Klainerman, Machedon, Foschi, Selberg, and others. Previously the only results were for $p \geq 2$.
- We can improve the "1/8" square function estimate of Mockenhaupt for the cone; this leads to some improvements on the local smoothing conjecture, the Bochner-Riesz conjecture for the cone, and the smoothing conjecture for convolution with the helix.