Mathematics 245B
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Instructions: Try to do all three problems; they are all of equal value. There is plenty of working space, and a blank page at the end.

You may freely use the axiom of choice in your work, and cite any theorem or result from the class notes or textbook.

Good luck!

Name: ______________________________

Student ID: ______________________________

Signature: ______________________________

Problem 1. _______________

Problem 2. _______________

Problem 3. _______________

Total: ______________________________

Problem 1. Let $1 \leq p < \infty$, and let $f \in L^p(\mathbf{R})$, where we give $\mathbf{R}$ the usual Lebesgue measure and σ-algebra. For every $t \in \mathbf{R}$, let $f_t \in L^p(\mathbf{R})$ be the translated function $f_t(x) := f(x-t)$. Show that f_t converges in L^p norm to f in the limit $t \to 0$ (i.e. $\lim_{t\to 0} \|f_t - f\|_{L^p(\mathbf{R})} = 0$). (*Hint:* First verify this when f is a step function $1_{[a,b]}$, then when f is an indicator function 1_E of some set of finite measure E, then as a simple function, then as an arbitrary L^p function. Alternatively, show that the space of $f \in L^p(\mathbf{R})$ which obeys the desired properties is a closed linear subspace of L^p that contains the step functions.)

Let V be the space of all $f \in L^p(\mathbf{R})$ such that $\lim_{t\to 0} \|f_t - f\|_{L^p(\mathbf{R})} = 0$, thus V is a subset of $L^p(\mathbf{R})$. It is closed under addition: if $f, g \in V$, then $f + g \in V$, because $\|(f+g)_t - (f+g)\|_{L^p(\mathbf{R})} \leq \|f_t - f\|_{L^p(\mathbf{R})} + \|g_t - g\|_{L^p(\mathbf{R})}$ thanks to the triangle inequality (and the fact that translation $f \mapsto f_t$ is a linear operation). For similar reasons, it is closed under multiplication: if $c \in \mathbf{C}$ and $f \in V$, then $cf \in V$, because $\|(cf)_t - cf\|_{L^p(\mathbf{R})} = |c|\|f_t - f\|_{L^p(\mathbf{R})}$. Thus V is a vector space.

Also observe that V is closed in $L^p(\mathbf{R})$: if $f^{(n)} \in V$ converges in $L^p(\mathbf{R})$ to f, then for any $\varepsilon > 0$ we have $\|f^{(n)} - f\|_{L^p(\mathbf{R})} \leq \varepsilon$ for some sufficiently large n. Since translation does not affect Lebesgue measure (and thus does not affect the L^p norm), we also have $\|f^{(n)}_t - f_t\|_{L^p(\mathbf{R})} \leq \varepsilon$ for all $t \in \mathbf{R}$ (it is important here that the RHS is uniform in t). Fixing n, we then see that as $f^{(n)} \in V$, we also have $\|f^{(n)}_t - f^{(n)}\|_{L^p(\mathbf{R})} \leq \varepsilon$ for all sufficiently small t. Putting this all together using the triangle inequality, we see that $\|f_t - f\|_{L^p(\mathbf{R})} \leq 3\varepsilon$ for all sufficiently small t, and thus $f \in V$.

[One can also deduce the closed nature of V from the more general fact that the uniform limit of continuous functions from some topological space to some metric space is again continuous, but this is not much shorter than (or particularly different from) the above direct proof.]

On the other hand, if $f = 1_{[a,b]}$ is the indicator function of an interval, then we see that f_t converges pointwise almost everywhere to f, and is also dominated by an L^p function for t bounded, so by dominated convergence we see that $f_t \to f$ in L^p as $t \to 0$. (One can of course also work out the integrals by hand; note that the cases $t > 0$ and $t < 0$ are slightly different, as are the cases $|t| < b - a$ and $|t| \geq b - a$ (though the latter is irrelevant in the limit $t \to 0$). Note also that for general f, it is not the case that f_t converges pointwise almost everywhere to f; this is true if and only if f is continuous almost everywhere.) Thus the indicator function of an interval lies in V. By linearity, this means that the indicator function of any finite union of intervals lie in V. Since every measurable set $E \subset \mathbf{R}$ of finite measure can be approximated to arbitrary accuracy (in Lebesgue measure) by a finite union of intervals, every indicator function 1_E can be approximated in $L^p(\mathbf{R})$ norm to arbitrary accuracy by the indicator function of a finite union of intervals. Since V is closed, we thus see that every indicator function with finite measure support lies in V; by linearity, this means that any simple function with finite measure support lies in V. Since such functions are dense in $L^p(\mathbf{R})$, V must equal all of $L^p(\mathbf{R})$, and we are done.

Note that the claim fails when $p = \infty$; consider for instance what happens with $f = 1_{[0,1]}$. (Indeed, f_t converges to f in $L^\infty(\mathbf{R})$ if and only if f is (essentially) uniformly continuous (i.e.

uniformly continuous after modification on a set of measure zero).)

A shorter proof would be to observe that the continuous functions of compact support lie in V (this follows either from dominated convergence or uniform continuity), and that these are dense in $L^p(\mathbf{R})$, which can be proven by a variety of methods.

Problem 2. Let $([0,1],\mathcal{B})$ be the unit interval with the Borel σ-algebra. Let $M([0,1])$ be the space of real finite measures $\mu : \mathcal{B} \to \mathbf{R}$, with the norm $\|\mu\| := |\mu|([0,1])$. Show that $M([0,1])$ is a real Banach space. (You may assume without proof that $M([0,1])$ is a normed vector space.)

It suffices to show that every absolutely convergent series $\sum_{n=1}^{\infty} \mu_n$ in $M([0,1])$ is also conditionally convergent.

For any Borel set E, we have $|\mu_n(E)| \leq |\mu_n|(E) \leq \|\mu_n\|$. Since $\sum_{n=1}^{\infty} \|\mu_n\|$ is absolutely convergent, we see that $\sum_{n=1}^{\infty} \mu_n(E)$ is also. Write $\mu(E) := \sum_{n=1}^{\infty} \mu_n(E)$ for this limit.

Clearly $\mu(\emptyset) = 0$. Also, we have the finiteness bound $|\mu(E)| \leq \sum_{n=1}^{\infty} \|\mu_n\| < \infty$ by the above discussion. Finally, we claim that μ is countably additive (and is hence a real finite measure). To see this, let $E_1, E_2, \ldots$ be disjoint measurable sets. Observe that $\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} |\mu_n(E_m)| \leq \sum_{n=1}^{\infty} \|\mu_n\| < \infty$. Thus by Fubini's theorem, we have

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mu_n(E_m) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \mu_n(E_m)$$

(in particular, all series here are absolutely convergent). But the left-hand side is $\mu(\bigcup_{m=1}^{\infty} E_m)$ and the right-hand side is $\sum_{m=1}^{\infty} \mu(E_m)$, and so we obtain countable additivity.

It remains to show that $\sum_{n=1}^{N} \mu_n$ converges in norm to μ. Observe for any set E that

$$|\mu(E) - \sum_{n=1}^{N} \mu_n(E)| = |\sum_{n=N+1}^{\infty} \mu_n(E)| \leq \sum_{n=N+1}^{\infty} \|\mu_n\|$$

and thus

$$\|\mu - \sum_{n=1}^{N} \mu_n\| \leq \sum_{n=N+1}^{\infty} \|\mu_n\|.$$

Letting $N \to \infty$ we obtain the claim.

An alternate (less elementary) proof is as follows. It suffices to prove every Cauchy sequence converges. If μ_n is a Cauchy sequence, then it is bounded. If we now write $\nu := \sum_{n=1}^{\infty} 2^{-n}|\mu_n|$, then ν is an unsigned finite measure (by the above arguments), and the μ_n are all absolutely continuous with respect to ν, thus by the Radon-Nikodym theorem $d\mu_n = f_n d\nu$ for some $f_n \in L^1(\nu)$. Since the μ_n are Cauchy in $M([0,1])$, the f_n are Cauchy in $L^1(\nu)$, and hence by completeness of $L^1(\nu)$ converge to some $f \in L^1(\nu)$. One then verifies that μ_n converges to the measure μ defined by $d\mu = f d\nu$.

Problem 3. Show that there exists a continuous linear functional $\mu : \ell^\infty(\mathbf{Z}) \to \mathbf{C}$ with the property that $\mu(1) = 1$ (where the $1 \in \ell^\infty(\mathbf{Z})$ on the left-hand side represents the constant function 1 on the integers $\mathbf{Z}$), and such that one has the translation invariance property $\mu(f_m) = \mu(f)$ for all $m \in \mathbf{Z}$ and $f \in \ell^\infty(\mathbf{Z})$, where $f_m(n) := f(n-m)$ is the translation of f by m. (Hint: using the Hahn-Banach theorem, one can create a generalised limit functional $\lambda : \ell^\infty(\mathbf{N}) \to \mathbf{C}$ that extends the usual limit function; then set $\mu(f) := \lambda((\frac{1}{2n+1}\sum_{j=-n}^n f(j))_{n\in\mathbf{N}})$ to be the generalised limit of the averages $(\frac{1}{2n+1}\sum_{j=-n}^n f(j))_{n\in\mathbf{N}}$.)

Let $\lim : c(\mathbf{N}) \to \mathbf{C}$ be the limit functional $(a_n)_{n=1}^\infty \to \lim_{n\to\infty} a_n$ on the space $c(\mathbf{N}) \subset \ell^\infty(\mathbf{N})$ of convergent sequences, equipped with the sup norm. This is a bounded linear functional, and hence by Hahn-Banach has an extension $\lambda : \ell^\infty(\mathbf{N}) \to \mathbf{C}$ to $\ell^\infty(\mathbf{N})$, which is also a bounded linear functional.

Now define $\mu(f) := \lambda((\frac{1}{2n+1}\sum_{j=-n}^n f(j))_{n\in\mathbf{N}})$. Observe that if f is a sequence bounded in magnitude by M (say), then the averages $\frac{1}{2n+1}\sum_{j=-n}^n f(j)$ are also bounded in magnitude by M, and so μ maps ℓ^∞ to $\mathbf{C}$ and is bounded; it is also clearly linear. Since $\lambda(1) = 1$, and the averages of the sequence $f(j) := 1$ are also 1, we see that $\mu(1) = 1$. Finally, for any m, observe that

$$\mu(f) - \mu(f_m) = \lambda((\frac{1}{2n+1}\sum_{j=-n}^n f(j) - \frac{1}{2n+1}\sum_{j=-n}^n f(j-m))_{n\in\mathbf{N}}).$$

For $n > |m|$, a comparision of the two summations reveals that they differ in exactly $2|m|$ terms. Thus, if f is bounded in magnitude by M,

$$|\frac{1}{2n+1}\sum_{j=-n}^n f(j) - \frac{1}{2n+1}\sum_{j=-n}^n f(j-m)| \leq \frac{2|m|}{2n+1},$$

so in particular the expression inside the absolute value goes to zero as $n \to \infty$ (keeping m fixed). Since λ extends the limit functional, we conclude that

$$\lambda((\frac{1}{2n+1}\sum_{j=-n}^n f(j) - \frac{1}{2n+1}\sum_{j=-n}^n f(j-m))_{n\in\mathbf{N}}) = 0,$$

and so $\mu(f) = \mu(f_m)$ as required.

(Note, incidentally, that to verify $\mu(f) = \mu(f_m)$, it suffices by induction and symmetry to do so in the case $m = 1$, although the case of general m is not really that much harder than the $m = 1$ case.)

The function μ is known as an *invariant mean* on $\mathbf{Z}$, and groups with an invariant mean are known as *amenable groups*. They play an important role in harmonic analysis, geometric group theory, and representation theory (and have already shown up in the notes on the Banach-Tarski paradox).
