Mathematics 245B
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Final, March 24, 2005

Instructions: Do all seven problems; they are all of equal value. There is plenty of working space, and a blank page at the end.

You may enter in a nickname if you want your final score posted. Good luck!

Name: ____________________

Nickname: ____________________

Student ID: ____________________

Signature: ____________________

Problem 1 (10 points). ____________

Problem 2 (10 points). ____________

Problem 3 (10 points). ____________

Problem 4 (10 points). ____________

Problem 5 (10 points). ____________

Problem 6 (10 points). ____________

Problem 7 (10 points). ____________

Total (70 points): ____________________

Problem 1. Let H be a Hilbert space, and let M be a non-trivial closed subspace of H (i.e. $M \neq \{0\}$. Let $P_M : H \to H$ be the orthogonal projection from H to M, thus in the notation of Theorem 5.24, we have $P_M x = y \in M$ and $x - P_M x = z \in M^\perp$ for all $x \in H$. You may assume without proof that P_M is linear. Show that $\|P_M\|_{op} = 1$ (so in particular P_M is bounded), that $P_M^2 = P_M$, and (using the notation of Q57 from Chapter 5) $P_M = P_M^*$. (More succinctly, orthogonal projections are linear idempotent self-adjoint contractions).

Since $P_M x$ lies in M and $x - P_M x$ lies in $M^\perp$, we see that $P_M x$ and $x - P_M x$ are orthogonal, hence by Pythagoras's theorem

$$\|x\|^2 = \|P_M x\|^2 + \|x - P_M x\|^2.$$

In particular we have $\|P_M x\| \leq \|x\|$, which implies that $\|P_M\|_{op} \leq 1$. Also, if x is any non-zero element of M (which exists since M is non-trivial) we see that $P_M x = x$, and hence $\|P_M x\| = \|x\|$ for this vector. Thus $\|P_M\|_{op} \geq 1$, which when combined with the preceding gives $\|P_M\|_{op} = 1$.

If $x \in H$, then $P_M x \in M$, and then $P_M(P_M x) = P_M x$ (the orthogonal projection of M onto M is the identity). Thus $P_M^2 = P_M$.

Finally, we show self-adjointness. By definition of adjoint, we have

$$\langle P_M x, y \rangle = \langle x, P_M^* y \rangle \text{ for all } x, y \in H.$$

Applying this for all $x \in M^\perp$ we see that

$$\langle x, P_M^* y \rangle = 0 \text{ for all } x \in M^\perp \text{ and } y \in H$$

and thus $P_M^* y \in (M^\perp)^\perp = M$ for all $y \in H$. Also, applying the previous identity for all $x \in M$ gives

$$\langle x, y \rangle = \langle x, P_M^* y \rangle \text{ for all } x \in M \text{ and } y \in H$$

and hence $y - P_M^* y \in M^\perp$ for all $y \in H$. Combining these two and recalling the uniqueness of orthogonal projection we obtain $P_M^* y = P_M y$ for all $y \in H$, and hence $P_M^* = P_M$ as desired.

Problem 2. Let H be a Hilbert space, and let $P : H \to H$ be a continuous linear mapping such that $P^2 = P$ and $P = P^*$ (using the notation from Q57 from Chapter 5). Show that there exists a closed linear subspace M of H such that $P = P_M$, where P_M is the orthogonal projection operator defined in Problem 1. (Remark: this shows that all linear idempotent self-adjoint contractions are orthogonal projections, thus providing a converse to Problem 1). Hint: analyze the behavior of P on the range $P(H)$ and on the cokernel $P(H)^\perp$.

If $x \in P(H)$, then $x = Py$ for some $y \in H$, and $Px = P^2 y = Py = x$. Thus P is the identity on $P(H)$.

Now suppose that $x \in P(H)^\perp$. Then we have $\langle x, Py \rangle = 0$ for all $y \in H$. By self-adjointness this means that $\langle Px, y \rangle = 0$ for all $y \in H$, hence $Px \in H^\perp = \{0\}$. Thus P is zero on $P(H)^\perp$. Now let M be the closure of $P(H)$. Then by continuity we see that P is the identity on M, and is zero on $P(H)^\perp = M^\perp$. In particular,

$$Px = P(P_M x + (x - P_M x)) = P(P_M x) + P(x - P_M x) = P_M x + 0 = P_M x$$

and hence $P = P_M$ as desired.

Problem 3. Let $(X, (\|\|_n)_{n=1}^\infty)$ be a Frechet space equipped with a countable number of semi-norms $\|\|_1, \|\|_2, \ldots$. Let $d : X \times X \to \mathbf{R}^+$ be the function

$$d(x,y) := \sum_{n=1}^{\infty} 2^{-n} \frac{\|x-y\|_n}{1+\|x-y\|_n}.$$

You may assume without proof that this series is convergent, and that d is a metric. Show that the topology generated by the metric space (X,d) is the same as the topology generated by the Frechet space $(X, (\|\|_n)_{n=1}^\infty)$, and show that (X,d) is complete. (Note that the latter claim does not automatically follow from the former, because completeness is not a topological property).

We need to show that every open set in the Frechet space is also open in the metric space, and vice versa. Let us first show that every open space in the Frechet space is open in the metric space. Since the Frechet space topology is generated by the sub-basic sets

$$B_n(x_0, r) = \{x \in X : \|x - x_0\|_n < r\},$$

it suffices to show that these sets are also open in the metric topology, or equivalently that

$$X \backslash B_n(x_0, r) = \{x \in X : \|x - x_0\|_n \geq r\}$$

is closed in the metric topology. Thus, let x_m be a sequence in $X \backslash B_n(x_0, r)$ which converges to some point y in the metric d, thus $d(x_m, y) \to 0$. By definition of d this implies that $\frac{\|x_m - y\|_n}{1+\|x_m-y\|_n} \to 0$, which by continuity of $\frac{x}{1+x}$ and its inverse, implies that $\|x_m - y\|_n \to 0$, and hence by the triangle inequality $\|y - y_0\|_n \geq r$, as desired.

Conversely, let us show that every open set in the metric space is also open in the Frechet space. It suffices to show that a metric ball

$$B(x_0, r) = \{x \in X : d(x, x_0) < r\}$$

is open in the Frechet space. Pick $x \in B(x_0, r)$, then there exists an integer N such that $d(x, x_0) < r - 2^{-N+1}$. Now we claim that $B(x_0, r)$ contains the basic open set

$$\bigcap_{n=1}^{N} B(x, 2^{-N}).$$

Indeed, if y lies in the above set, then $\|x - y\|_n \leq 2^{-N}$ for all $1 \leq n \leq N$, and hence

$$d(x,y) \leq \sum_{n=1}^{N} 2^{-n} 2^{-N} + \sum_{n=N+1}^{\infty} 2^{-n} \leq 2^{-N} + 2^{-N} = 2^{-N+1}$$

and hence by the triangle inequality $d(y, x_0) < r$, as desired.

Problem 4. Let X be a real vector space. Let $(\mathcal{F}_\alpha)_{\alpha \in A}$ be a family of topologies on X, such that for each $\alpha \in A$, $(X, \mathcal{F}_\alpha)$ is a topological vector space (i.e. addition and scalar multiplication are continuous, see Sec 5.4). Note that A could be uncountable. Let $\mathcal{F} := \bigvee_{\alpha \in A} \mathcal{F}_\alpha$ be the topology generated by $\bigcup_{\alpha \in A} \mathcal{F}_\alpha$ (thus $\bigcup_{\alpha \in A} \mathcal{F}_\alpha$ is a sub-base for $\mathcal{F}$). Show that $(X, \mathcal{F})$ is also a topological vector space.

We need to show that the addition map $+ : X \times X \to X$ and the scalar multiplication map $\cdot : \mathbf{R} \times X \to X$ are continuous. We begin with addition. We need to show that the inverse image $+^{-1}(V) = \{(x, y) \in X \times X : x + y \in V\}$ of any open set in $\mathcal{F}$ is still open. Actually we only need to check this for the sub-basic open sets, thus sets in $\mathcal{F}_\alpha$. But the inverse images of those sets will be open in $\mathcal{F}_\alpha \times \mathcal{F}_\alpha$, and hence also open in the larger topology $\mathcal{F} \times \mathcal{F}$. A similar argument also works for scalar multiplication.

Problem 5. Let $C([0,1])$ be the space of continuous real-valued functions $f:[0,1]\to\mathbf{R}$, equipped with the uniform norm $\|f\|_\infty := \sup_{x\in[0,1]}|f(x)|$. (You may assume without proof that $C[0,1])$ is a Banach space). If $f\in C([0,1])$, and $M,r>0$ are positive numbers, show that the sets

$$E_{M,r} := \{f\in C([0,1]) : \text{There exists } x\in\mathbf{R} \text{ such that } |f(x+h)-f(x)|\leq M|h| \text{ for all } |h|\leq r\}$$

is closed in $C([0,1])$ and contains no interior points. (Hint: you may find the "zigzag" function

$$f_\varepsilon(x) := \inf\{|x-\varepsilon n| : n\in\mathbf{Z}\}$$

to be useful). Using the Baire category theorem (Theorem 5.9), conclude that there exists a function $f\in C([0,1])$ which is nowhere differentiable (i.e. $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ does not exist for any $x\in\mathbf{R}$).

(Note: the version of this question in the final was severely broken and had to be discarded). Closure is easy: if f_n is a sequence in $E_{M,r}$ which converges uniformly to a function $f\in C([0,1])$, then by taking limits of the inequality $|f_n(x+h)-f_n(x)|\leq M|h|$ we obtain $|f(x+h)-f(x)|\leq M|h|$, and hence $f\in E_{M,r}$. Now we show that $E_{M,r}$ has no interior. Let f be an element of $E_{M,r}$, and let $\varepsilon>0$ be arbitrary. Using the Weierstrass approximation theorem, we can find a polynomial P such that $\|f-P\|_\infty\leq\varepsilon$. Note that P' is also continuous on $[0,1]$, hence is bounded, which means that P is Lipschitz with some large Lipschitz constant L. Now we consider the function $g := P + Nf_{2\varepsilon/N}$ for some very large N. One can verify that $\|Nf_{\varepsilon/N}\|_\infty \leq N\frac{\varepsilon}{N}\leq\varepsilon$, so by the triangle inequality we have $\|f-g\|_\infty\leq 2\varepsilon$. Also, if N is large enough (bigger than $L+M$ will suffice) then one can easily check that g cannot lie in $E_{M,r}$ (it always has slope bigger than $N-L$ or less than $-N+L$). Thus f is arbitrarily close to an element outside of $E_{M,r}$, and hence $E_{M,r}$ has no interior.

The sets $E_{M,r}$ are thus nowhere dense, and so by the Baire category theory $\bigcup_{M\in\mathbf{N}}\bigcup_{R\in\mathbf{N}}E_{M,1/R}$ cannot cover all of $C([0,1])$. Hence there exists a function f which does not lie in $E_{M,1/R}$ for any positive integer M,R. But by definition of $E_{M,1/R}$ this forces f to be nowhere differentiable.

Problem 6. Let $n \geq 1$ be an integer, let K be a compact subset of $\mathbf{R}^n$, and let $L^1(K)$ be the space of absolutely integrable functions on K, with the norm $\|f\|_1 := \int_K |f(x)|\,dx$, and two functions f and g identified if they agree almost everywhere. You may assume without proof that $L^1(K)$ is a Banach space. Let $P(K)$ be those functions $f \in L^1(K)$ (or more precisely, equivalence classes of functions) which are polynomials, thus one has

$$f(x_1, \ldots, x_n) = \sum_{0 \leq i_1, \ldots, i_n \leq N} c_{i_1,\ldots,i_n} x_1^{i_1} \ldots x_n^{i_n}$$

for some constants $c_{i_1,\ldots,i_n}$ and some integer N, and all $(x_1, \ldots, x_n) \in K$. Show that $P(K)$ is dense in $L^1(K)$. (Hint: you will find the Stone-Weierstrass theorem (Theorem 4.45), Urysohn's lemma (Lemma 4.15), Theorem 2.26, and Theorem 2.40 to be useful).

We first observe that $P(K)$ is dense in $C(K)$ (with the uniform topology). This follows from the Stone-Weierstrass theorem, since $P(K)$ is an algebra which separates points (as can be seen from the monomial polynomials $x_1, \ldots, x_n$) and contains the identity 1. In particular $P(K)$ also is dense in $C(K)$ in the L^1 topology. So it will suffice to show that $C(K)$ is dense in $L^1(K)$.

Let us first show that for any measurable subset E of K, the set 1_E lies in the closure of $C(K)$. Pick any $\varepsilon > 0$. By Theorem 2.40(a), one can find an open set V and a compact set K' such that $K' \subset E \subset V$ and $m(V \backslash K') \leq \varepsilon$. Using Urysohn's lemma, one can find a function $f \in C(K)$ which equals 1 on K' and equals zero outside of V, and take values between 0 and 1 elsewhere, and thus $\|f - 1_E\|_1 \leq \varepsilon$. This shows that 1_E is adherent to $C(K)$, and thus lies in its closure. Taking linear combinations we see that all simple functions lie in the closure of $C(K)$; using Theorem 2.26 we conclude then that the closure of $C(K)$ is in fact all of $L^1(K)$, as desired.

Problem 7. Let X be a real vector space (with no topology!), and let B be a convex subset of X, with the property that the intersection of B with any line through the origin is a symmetric open interval; more precisely, for any non-zero $x \in X$ there exists an $M(x) > 0$ such that

$$\{t \in \mathbf{R} : tx \in B\} = \{t \in \mathbf{R} : -M(x) < t < M(x)\}.$$

Let x_0 be an element of X which does not lie in B. Show that there exists a linear functional $\lambda : X \to \mathbf{R}$ (not necessarily continuous!) such that $\lambda(x_0) \geq 1$, and $\lambda(x) < 1$ for all $x \in B$. (This is a special case of the "Separating Hyperplane Theorem"). Hint: show that the function $\|x\|_B := 1/M(x)$ is a norm, then use Theorem 5.8(b).

We let $\|x\|_B := 1/M(x)$ when $x \neq 0$, and define $\|0\|_B := 0$. This norm is clearly non-negative, non-degenerate, and homogeneous. Now we verify the triangle inequality $\|x+y\|_B \leq \|x\|_B + \|y\|_B$. This is easy if x, y, or $x+y$ are zero, so suppose that they are all non-zero. Observe that $tx \in B$ for all $0 < t < 1/\|x\|_B$ and $sy \in B$ for all $0 < s < 1/\|y\|_B$, so by convexity of B, $\frac{s}{s+t}tx + \frac{t}{s+t}sy \in B$ for all such s, t. In other words, $\frac{1}{1/s+1/t}(x+y) \in B$ whenever $1/t > \|x\|_B$ and $1/s > \|y\|_B$, which means that $\frac{1}{r}(x+y) \in B$ whenever $r > \|x\|_B + \|y\|_B$. This forces $M(x+y) \geq \frac{1}{\|x\|_B+\|y\|_B}$ and hence $\|x+y\|_B \leq \|x\|_B + \|y\|_B$ as desired.

Since x_0 does not lie in B, we have $M(x_0) \leq 1$, and hence $\|x_0\|_B \geq 1$. Now by Theorem 5.8(b) we can find a linear functional $\lambda : X \to \mathbf{R}$ with $\|\lambda\|_{X^*} = 1$ such that $\lambda(x_0) = \|x_0\|_B \geq 1$. In particular, if $x \in B$, then $M(x) > 1$, thus $\|x\|_B < 1$, and then $|\lambda(x)| < 1$, and the claim follows.

This blank page can be used as additional space for your answers (but please indicate which question is being answered!).