Mathematics 245B
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Midterm, Feb 11, 2003

Instructions: Try to do all three problems; they are all of equal value. There is plenty of working space, and a blank page at the end.

You may enter in a nickname if you want your midterm score posted.

Good luck!

Name: ______________________

Nickame: ______________________

Student ID: ______________________

Signature: ______________________

Problem 1. ____________

Problem 2. ____________

Problem 3. ____________

Total: ______________________

Problem 1. (a) Let X be a compact topological space, and let $\mathcal{F} \subset C(X; \mathbf{R})$ be a collection of continuous functions from X to $\mathbf{R}$ which is pointwise bounded (i.e. the set $\{f(x) : f \in \mathcal{F}\}$ is bounded for all $x \in X$) and equicontinuous. Prove that $\mathcal{F}$ is uniformly bounded (i.e. there exists an $M > 0$ such that $|f(x)| \leq M$ for all $x \in X$ and $f \in \mathcal{F}$).

(b) Given an example to show that the results in part (a) fail if the assumption of equicontinuity is dropped.

Problem 2. (a) Let X, Y be topological spaces, and suppose that Y is Hausdorff. Let A be a dense subset of X, and let $f : X \to Y$ and $g : X \to Y$ be continuous functions. Show that if f and g agree on A (i.e. $f(x) = g(x)$ for all $x \in A$), then they agree on all of X (i.e. f and g are identical).

(b) Now suppose that X is normal, and A is a subset of X which is *not* dense. Show that there exist continuous functions $f : X \to \mathbf{R}$ and $g : X \to \mathbf{R}$ which agree on A but do not agree on all of X.

Problem 3. Let X be a Banach space, and let $l^1(\mathbf{Z})$ be the space of absolutely summable sequences $(x_n)_{n\in\mathbf{Z}}$, with the l^1 norm $\|(x_n)_{n\in\mathbf{Z}}\|_{l^1(\mathbf{Z})} := \sum_{n\in\mathbf{Z}} |x_n|$. For each integer n, let $e_n \in l^1(\mathbf{Z})$ be the element of $l^1(\mathbf{Z})$ whose n^{th} entry is 1 and whose other entries are zero (thus $(e_n)_m = 1$ if $m = n$ and $(e_n)_m = 0$ otherwise).

Let $T : l^1(\mathbf{Z}) \to X$ be a linear transformation. Show that T is continuous if and only if the set $\{T(e_n) : n \in \mathbf{Z}\}$ is a bounded subset of X.