Mathematics 245A
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Instructions: Do all seven problems; they are all of equal value. There is plenty of working space, and a blank page at the end.

You may enter in a nickname if you want your final score posted. Good luck!

Name: ______________________

Nickame: ______________________

Student ID: ______________________

Signature: ______________________

Problem 1 (10 points). ____________

Problem 2 (10 points). ____________

Problem 3 (10 points). ____________

Problem 4 (10 points). ____________

Problem 5 (10 points). ____________

Problem 6 (10 points). ____________

Problem 7 (10 points). ____________

Total (70 points): ______________________

Problem 1. Let (X, d) be a non-empty metric space. We say that a point $x \in X$ is a *limit point* if it is the limit of some sequence $x_1, x_2, \ldots$ in X such that $x_n \neq x$ for all n. We say that X is *perfect* if every point is a limit point.

Show that every metric space which is both complete and perfect, must be uncountable. (Hint: use the Baire category theorem).

(Note: the hypothesis that X was non-empty was mistakenly omitted from the print version of this exam).

Let $x \in X$. Since X is a metric space, the singleton set $\{x\}$ is closed. Since X is perfect, x is a limit point, and hence x is adherent to the complement $X \backslash \{x\}$ of $\{x\}$, and thus is not an interior point of $\{x\}$. Thus $\{x\}$ has no interior, and hence (being closed) is nowhere dense.

Since X is non-empty and complete, it cannot be the countable union of nowhere dense sets, by the Baire category theorem. Thus X cannot be a countable union of points, and thus must be uncountable.

An equivalent proof proceeds by observing that $X - \{x\}$ is open dense for every x, and hence every countable intersection of these sets is also dense.

Problem 2. Let X be a Banach space. Show that the weak and weak-* topologies on X^* co-incide if and only if X is reflexive.

The weak topology on X^* is generated by using the sets $\{\omega \in X^* : |\xi(\omega) - z| < r\}$ as sub-basic open sets, where ξ ranges over X^{**}, z is a complex number, and $r > 0$. In contrast, the weak-* topology is generated by the sets $\{\omega \in X^* : |\omega(x) - z| < r\}$ where x ranges over X, z is a complex number, and $r > 0$. These sets can also be rewritten as $\{\omega \in X^* : |\hat{x}(\omega) - z| < r\}$. Thus if X is reflexive, then the sub-basic sets are identical and thus the topologies are also identical.

Now suppose that X is not reflexive, then there exists $\xi \in X^{**}$ such that $\xi \neq \hat{x}$ for any $x \in X$. Let B be the set $\{\omega \in X^* : |\xi(\omega)| < 1\}$. Then B is open in the weak topology. We claim that B is not open in the weak-* topology, and therefore that the two topologies are distinct. To prove this, suppose for contradiction that B was open in the weak-* topology. In particular, B must contain a basic neighborhood of 0, i.e. there exist $x_1, \ldots, x_N$ and $r_1, \ldots, r_N$ such that

$$\bigcap_{i=1}^{N} \{\omega \in X^* : |\omega(x_n)| < r_n\} \subseteq \{\omega \in X^* : |\xi(\omega)| < 1\}.$$

In particular, we have

$$\bigcap_{i=1}^{N} \{\omega \in X^* : \omega(x_n) = 0\} \subseteq \{\omega \in X^* : |\xi(\omega)| < 1\}.$$

Multiplying this by ε, we obtain

$$\bigcap_{i=1}^{N} \{\omega \in X^* : \omega(x_n) = 0\} \subseteq \{\omega \in X^* : |\xi(\omega)| < \varepsilon\}$$

for any $\varepsilon > 0$, and hence

$$\bigcap_{i=1}^{N} \{\omega \in X^* : \omega(x_n) = 0\} \subseteq \{\omega \in X^* : \xi(\omega) = 0\}.$$

If one lets $\Phi : X^* \to \mathbf{C}^N$ be the linear map $\Phi(\omega) := (\omega(x_1), \ldots, \omega(x_N))$, then the above inclusion asserts that the kernel of Φ is contained in the kernel of ξ. This implies that $\xi = T \circ \Phi$ for some map $T : \mathbf{C}^N \to \mathbf{C}$. Since ξ and Φ are linear, then T is also. But this then means that ξ is a linear combination of the $\hat{x}_n$, and thus lies in the image of X, a contradiction.

Note: it is quite difficult to prove this assertion using convergent sequences (or more precisely, convergent nets, since weak topologies tend not to be first countable).

Problem 3. Let H be a Hilbert space, and let $(e_\alpha)_{\alpha \in A}$ be an orthonormal basis for H (which may be finite, countable, or uncountable). Let $x_1, x_2, x_3, \ldots$ be a sequence of vectors in H which are bounded (i.e. there exists an $M > 0$ such that $\|x_n\| \leq M$ for all n). Let x be another vector in H.

Show that the sequence x_n is weakly convergent to x if and only if we have $\lim_{n\to\infty} \langle x_n, e_\alpha \rangle = \langle x, e_\alpha \rangle$ for all $\alpha \in A$.

"only if": If x_n is weakly convergent to x, then $\phi(x_n) \to \phi(x)$ for all $\phi \in H^*$. In particular, for any e_α, we have $\langle x_n, e_\alpha \rangle \to \langle x, e_\alpha \rangle$, since the map $x \mapsto \langle x, e_\alpha \rangle$ is a bounded linear functional on H (by the Cauchy-Schwarz inequality).

"if": Suppose $\langle x_n, e_\alpha \rangle \to \langle x, e_\alpha \rangle$ for all α. Let $\phi \in H^*$; our task is to show that $\phi(x_n) \to \phi(x)$. By the Riesz representation theorem for H, we can find a $y \in H$ such that $\phi(x) = \langle x, y \rangle$, so our task is now to show that $\langle x_n, y \rangle \to \langle x, y \rangle$.

If one immediately expands x_n and y in the basis e_α one will find significant difficulty interchanging the limit and summation. To avoid this, we first approximate y by a *finite* linear combination of the e_α. Pick any $\varepsilon > 0$. Since $\sum_{\alpha \in A} \langle y, e_\alpha \rangle e_\alpha$ converges unconditionally (but not absolutely!) to y, we can find a finite set $F \subset A$ such that

$$\| \sum_{\alpha \in A} \langle y, e_\alpha \rangle e_\alpha - y \| \leq \varepsilon.$$

In particular, by Cauchy-Schwarz we have

$$|\langle x_n, \sum_{\alpha \in A} \langle y, e_\alpha \rangle e_\alpha \rangle - \langle x_n, y \rangle| \leq \varepsilon \|x_n\| \leq \varepsilon M$$

and similarly

$$|\langle x, \sum_{\alpha \in A} \langle y, e_\alpha \rangle e_\alpha \rangle - \langle x, y \rangle| \leq \varepsilon \|x\|$$

On the other hand, by interchanging the *finite* sum with the limit we know that

$$\langle x_n, \sum_{\alpha \in A} \langle y, e_\alpha \rangle e_\alpha \rangle \to \langle x, \sum_{\alpha \in A} \langle y, e_\alpha \rangle e_\alpha \rangle$$

and so for n sufficiently large (depending on ε)

$$|\langle x_n, \sum_{\alpha \in A} \langle y, e_\alpha \rangle e_\alpha \rangle - \langle x, \sum_{\alpha \in A} \langle y, e_\alpha \rangle e_\alpha \rangle| \leq \varepsilon.$$

Putting all this together using the triangle inequality, we obtain

$$|\langle x_n, y \rangle - \langle x, y \rangle| \leq \varepsilon (M + \|x\| + 1).$$

Since $\varepsilon > 0$ was arbitrary, the claim follows.

Problem 4. Let W be a vector space. Let A be a non-empty index set, and for each $\alpha \in A$, let V_α be a subspace of W which is equipped with a norm $\|\|_{V_\alpha}$. Suppose that for each α, the norm $\|\|_{V_\alpha}$ turns V_α into a Banach space. Also assume the following compatibility condition: if $\alpha, \beta \in A$ and a sequence $x_n \in V_\alpha \cap V_\beta$ converges to x in the V_α norm and to x' in the V_β norm, then $x = x'$. Define a new vector space U by

$$U := \{x \in \bigcap_{\alpha \in A} V_\alpha : \sum_{\alpha \in A} \|x\|_{V_\alpha} < \infty\}$$

and equip this space with the norm

$$\|x\|_U := \sum_{\alpha \in A} \|x\|_{V_\alpha}.$$

Show that $\|\|_U$ is indeed a norm, and this norm turns U into a Banach space.

I apologize for not including the compatibility condition in the print version, without which the problem is false. Another equivalent form of the compatibility condition is the assertion that each pair $V_\alpha \cap V_\beta$ is a Banach space with the norm $\|x\|_{V_\alpha} + \|x\|_{V_\beta}$. Yet another formulation is that there is a Frechet space topology on W which is weaker than each of the V_α topologies when restricted to V_α. (Frechet spaces have unique limits). Also, the requirement that A be non-empty was also omitted.

First we verify the norm property. The triangle inequality for U follows from the triangle inequalities for the individual V_α norms:

$$\|x + y\|_U = \sum_{\alpha \in A} \|x + y\|_{V_\alpha} \leq \sum_{\alpha \in A} \|x\|_{V_\alpha} + \|y\|_{V_\alpha} = \|x\|_U + \|y\|_U.$$

The homogeneity property is proven similarly:

$$\|\lambda x\|_U = \sum_{\alpha \in A} \|\lambda x\|_{V_\alpha} \leq \sum_{\alpha \in A} |\lambda| \|x\|_{V_\alpha} = |\lambda| \|x\|_U.$$

Similarly we have non-degeneracy: if $\|x\|_U = 0$, then $\sum_{\alpha \in A} \|x\|_{V_\alpha} = 0$, hence $\|x\|_{V_\alpha} = 0$ for at least one V_α, hence $x = 0$. This concludes the proof that $\|\|_U$ is a norm.

Now we show that U is a Banach space. It suffices to show that any series $\sum_{n=1}^\infty x_n$ which is absolutely convergent in U, thus $\sum_{n=1}^\infty \|x_n\|_U < \infty$, is also conditionally convergent in U. Since $\|x_n\|_{V_\alpha} \leq \|x_n\|_U$, we see that $\sum_{n=1}^\infty x_n$ is absolutely convergent in V_α, hence conditionally convergent in V_α to some limit. By the compatibility condition, this limit is not dependent on α, thus $\sum_{n=1}^\infty x_n = x$ in each of the V_α topologies. Now we need to show that $\sum_{n=1}^\infty x_n = x$ in the U topology. We have

$$\|x - \sum_{n=1}^N x_n\|_{V_\alpha} = \| \sum_{n=N+1}^\infty x_n\|_{V_\alpha} \leq \sum_{n=N+1}^\infty \|x_n\|_{V_\alpha}$$

for all α and N; summing in α and using Fubini's theorem for sums we obtain

$$\|x - \sum_{n=1}^{N} x_n\|_U \leq \sum_{n=N+1}^{\infty} \|x_n\|_U.$$

Taking $N \to \infty$ and using the convergence of $\sum_{n=1}^{\infty} \|x_n\|_U$, the claim follows.

Problem 5. Let H be a Hilbert space. Recall that if M is a closed subspace of H, then we can define a linear operator $P_M : H \to H$ by defining $P_M x$ to be the element of M such that $x - P_M x \in M^\perp$ (see Problem 58 of Chapter 5).

Suppose that we have a sequence $M_1, M_2, \ldots$ of closed subspaces of H such that $M_n \subseteq M_{n+1}$ for all n. Let $M_\infty := \overline{\bigcup_{n=1}^\infty M_n}$, i.e. we let M_∞ be the closure of the union of all the subspaces M_n. Show that P_{M_n} converges to P_{M_∞} in the strong operator topology (see page 169 of textbook).

Let $x \in H$; our task is to show that $P_{M_n} x$ converges to $P_{M_\infty} x$ as $n \to \infty$.

Proof A: Recall that $P_{M_\infty} x$ is the closest point in M_∞ in x. In particular, it is closer to x than $P_{M_n} x$, which lies in M_n and hence in M_∞. In other words,

$$\|x - P_{M_\infty} x\| \le \|x - P_{M_n} x\|.$$

On the other hand, since $P_{M_\infty} x$ lies in the closure of $\bigcup_{n=1}^\infty M_n$, for every ε there exists an integer n and an $x_n \in M_n$ such that $\|x_n - P_{M_\infty} x\| \le \varepsilon$. But since $P_{M_n} x$ is the closest point in M_n to x we have

$$\|x - P_{M_n} x\| \le \|x - x_n\| \le \|x - P_{M_\infty} x\| + \varepsilon$$

by the triangle inequality. Putting these two facts together we see that

$$\|x - P_{M_n} x\| \to \|x - P_{M_\infty} x\| \text{ as } n \to \infty.$$

Now observe that $P_{M_n} x - P_{M_\infty} x$ lies in M_∞ and is hence orthogonal to $x - P_{M_\infty} x$. By Pythagoras we thus have

$$\|x - P_{M_n} x\|^2 = \|x - P_{M_\infty} x\|^2 + \|P_{M_\infty} x - P_{M_n} x\|^2,$$

and combining this with the previous observation we see that

$$\|P_{M_\infty} x - P_{M_n} x\| \to 0 \text{ as } n \to \infty$$

as desired.

Proof B: (Uses Axiom of Choice) We endow M_1 with an orthonormal basis $(e_\alpha)_{\alpha \in A_1}$. Since $M_1 \subset M_2$, we can extend this to an orthonormal basis $(e_\alpha)_{\alpha \in A_2}$ of M_2 for some $A_2 \supset A_1$. We can continue in this manner, choosing a basis $(e_\alpha)_{\alpha \in A_n}$ of M_n for each n.

Now let $A = \bigcup_{n=1}^\infty A_n$. The vectors $(e_\alpha)_{\alpha \in A}$ are still orthonormal, and all lie in $\bigcup_{n=1}^\infty M_n$, and hence in M_∞. If they do not form a basis for M_∞, then there would be a non-zero vector $v \in M_\infty$ which is orthogonal to all of the e_α. But this implies that v is orthogonal to each of the M_n, and hence to $\bigcup_{n=1}^\infty M_n$, and hence to M_∞, which implies that v is orthogonal to itself, which is absurd. Thus $(e_\alpha)_{\alpha \in A}$ is an orthonormal basis for M_∞, and in particular the series $\sum_{x \in A} \langle x, e_\alpha \rangle e_\alpha$ is unconditionally convergent to $P_{M_\infty} x$. In particular $\sum_{x \in A_n} \langle x, e_\alpha \rangle e_\alpha = P_{M_n} x$ converges to $P_{M_\infty} x$, as desired.

Problem 6. Let $(X, \mathcal{M}, \mu)$ be a sigma-finite measure space, and let $1 < p < \infty$. Let $f : X \to \mathbf{C}$ be a measurable function. Show that f lives in weak L^p (i.e. $[f]_p < \infty$) if and only if there exists an $M > 0$ such that for every measurable set E of finite measure, the integral $\int_E |f| \, d\mu$ is absolutely convergent and obeys the estimate

$$\int_E |f| \, d\mu \leq M \mu(E)^{1/p'}.$$

(Hint: You may wish to estimate the distribution function $\lambda_{f|_E}$ of the restriction $f|_E$ of f to E in terms of the distribution function λ_f of f itself.)

Again, I apologize, as the hypothesis that X was sigma finite was omitted from the print version of this question.

"if": Let $\alpha > 0$ be arbitrary, and let E be an arbitrary finite measure subset of $\{x \in X : |f(x)| > \alpha\}$. By hypothesis we have

$$\int_E |f| \, d\mu \leq M \mu(E)^{1/p'}$$

but on the other hand we have

$$\int_E |f| \, d\mu \geq \int_E \alpha \, d\mu = \alpha \mu(E).$$

Combining the two we obtain

$$\mu(E) \leq M^p / \alpha^p$$

for all finite measure subsets of $\{x \in X : |f(x)| > \alpha\}$. Using the sigma finite hypothesis, we obtain

$$\mu(\{x \in X : |f(x)| > \alpha\}) \leq M^p / \alpha^p$$

which shows that f is in weak L^p.

"only if": Let E be a set of finite measure. Observe that

$$\lambda_{f|_E}(\alpha) := \mu\{x \in E : |f(x)| > \alpha\} \leq \mu\{x \in X : |f(x)| > \alpha\} \leq [f]_p^p / \alpha^p$$

and also

$$\lambda_{f|_E}(\alpha) := \mu\{x \in E : |f(x)| > \alpha\} \leq \mu(E).$$

Thus

$$\int_E |f| \, d\mu = \int_0^\infty \lambda_{f|_E}(\alpha) \, d\alpha \leq \int_0^\infty \min([f]_p^p / \alpha^p, \mu(E)) \, d\alpha.$$

The latter integral is finite (for $0 < \alpha < 1$ use the $\mu(E)$ bound; for $\alpha > 1$ use the $[f]_p^p / \alpha^p$ bound) and the claim follows.

Problem 7. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces, and let $1 < p, q < \infty$. Let $T : L^p(X) \to L^q(Y)$ be a continuous linear operator which has the form

$$Tf(y) = \int_X K(x, y) f(x) \, d\mu(x) \text{ for all simple functions } f \in L^p(X)$$

for some complex-valued, absolutely integrable function K on $X \times Y$. (The restriction to simple functions is to ensure that the above integral is actually convergent for almost every y, thanks to Fubini's theorem). Let $T^\dagger : L^{q'}(Y) \to L^{p'}(X)$ be the adjoint of T as defined in Exercise 22 of Section 5.2, and where we have identified the dual of $L^p(X)$ with $L^{p'}(X)$, and the dual of $L^q(Y)$ with $L^{q'}(Y)$ in the usual manner (cf. Theorem 6.15). Show that

$$T^\dagger g(x) = \int_Y \overline{K(x, y)} g(y) \, d\nu(y) \text{ for all simple functions } g \in L^{q'}(Y).$$

Let g be a simple function, and hence in $L^{q'}(Y)$. If we think of $T^\dagger(g)$ as an element of the dual of $L^p(X)$, then it acts on functions $f \in L^p(X)$ by the formula

$$T^\dagger(g)(f) = g \circ Tf = \int_Y Tf(y)\overline{g(y)} \, dy$$

where we are identifying $g \in L^{q'}(Y)$ with a linear functional on $L^q(Y)$ in the usual manner. In particular, for simple functions f we have

$$T^\dagger(g)(f) = \int_Y \int_X K(x, y) f(x) \, d\mu(x) \overline{g(y)} \, d\nu(y).$$

By Fubini's theorem we then have

$$T^\dagger(g)(f) = \int_X f(x) \overline{\int_Y \overline{K(x, y)} g(y) \, d\nu(y)} d\mu(x).$$

Thus if we write $h(x) = \int_Y \overline{K(x, y)} g(y) \, d\nu(y)$, we see that

$$T^\dagger(g)(f) = \int_X f(x)\overline{h(x)} \, d\mu(x) \text{ for all simple } f.$$

By hypothesis, $T^\dagger(g)$ is a bounded linear functional on $L^p(X)$, hence $h \in L^{p'}(X)$. Now take limits using the fact that the simple functions are dense in L^p, to conclude that

$$T^\dagger(g)(f) = \int_X f(x)\overline{h(x)} \, d\mu(x) \text{ for all } f \in L^p(X).$$

Thus $T^\dagger(g)$ can be identified with h, and the claim follows.
