Mathematics 245A
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Instructions: Do all seven problems; they are all of equal value. There is plenty of working space, and a blank page at the end.

You may enter in a nickname if you want your final score posted. Good luck!

Name: ____________________

Nickame: ____________________

Student ID: ____________________

Signature: ____________________

Problem 1 (10 points). __________

Problem 2 (10 points). __________

Problem 3 (10 points). __________

Problem 4 (10 points). __________

Problem 5 (10 points). __________

Problem 6 (10 points). __________

Problem 7 (10 points). __________

Total (70 points): ____________________

Problem 1. Let (X, d) be a metric space. We say that a point $x \in X$ is a *limit point* if it is the limit of some sequence $x_1, x_2, \ldots$ in X such that $x_n \neq x$ for all n. We say that X is *perfect* if every point is a limit point.

Show that every metric space which is both complete and perfect, must be uncountable. (Hint: use the Baire category theorem).

Problem 2. Let X be a Banach space. Show that the weak and weak-* topologies on X^* coincide if and only if X is reflexive. (One direction is easy; the other requires the Hahn-Banach theorem).

Problem 3. Let H be a Hilbert space, and let $(e_\alpha)_{\alpha \in A}$ be an orthonormal basis for H (which may be finite, countable, or uncountable). Let $x_1, x_2, x_3, \ldots$ be a sequence of vectors in H which are bounded (i.e. there exists an $M > 0$ such that $\|x_n\| \leq M$ for all n. Let x be another vector in H.

Show that the sequence x_n is weakly convergent to x if and only if we have $\lim_{n\to\infty}\langle x_n, e_\alpha\rangle = \langle x, e_\alpha\rangle$ for all $\alpha \in A$.

Problem 4. Let W be a vector space. Let A be an index set, and for each $\alpha \in A$, let V_α be a subspace of W which is equipped with a norm $\|\|_{V_\alpha}$. Suppose that for each α, the norm $\|\|_{V_\alpha}$ turns V_α into a Banach space. Define a new vector space U by

$$U := \{x \in \bigcap_{\alpha \in A} V_\alpha : \sum_{\alpha \in A} \|x\|_{V_\alpha} < \infty\}$$

and equip this space with the norm

$$\|x\|_U := \sum_{\alpha \in A} \|x\|_{V_\alpha}.$$

Show that $\|\|_U$ is indeed a norm, and this norm turns U into a Banach space.

Problem 5. Let H be a Hilbert space. Recall that if M is a closed subspace of H, then we can define a linear operator $P_M : H \to H$ by defining $P_M x$ to be the element of M such that $x - P_M x \in M^\perp$ (see Problem 58 of Chapter 5).

Suppose that we have a sequence $M_1, M_2, \ldots$ of closed subspaces of H such that $M_n \subseteq M_{n+1}$ for all n. Let $M_\infty := \overline{\bigcup_{n=1}^\infty M_n}$, i.e. we let M_∞ be the closure of the union of all the subspaces M_n. Show that P_{M_n} converges to P_M in the strong operator topology (see page 169 of textbook).

Problem 6. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $1 < p < \infty$. Let $f : X \to \mathbf{C}$ be a measurable function. Show that f lives in weak L^p (i.e. $[f]_p < \infty$) if and only if there exists an $M > 0$ such that for every measurable set E of finite measure, the integral $\int_E |f| \, d\mu$ is absolutely convergent and obeys the estimate

$$\int_E |f| \, d\mu \le M \mu(E)^{1/p'}.$$

(Hint: You may wish to estimate the distribution function $\lambda_{f|_E}$ of the restriction $f|_E$ of f to E in terms of the distribution function λ_f of f itself.)

Problem 7. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces, and let $1 < p, q < \infty$. Let $T : L^p(X) \to L^q(Y)$ be a continuous linear operator which has the form

$$Tf(y) = \int_X K(x, y) f(x)\ d\mu(x) \text{ for all simple functions } f$$

for some complex-valued, absolutely integrable function K on $X \times Y$. (The restriction to simple functions is to ensure that the above integral is actually convergent for almost every y, thanks to Fubini's theorem). Let $T^\dagger : L^{q'}(Y) \to L^{p'}(X)$ be the adjoint of T as defined in Exercise 22 of Section 5.2, and where we have identified the dual of $L^p(X)$ with $L^{p'}(X)$, and the dual of $L^q(Y)$ with $L^{q'}(Y)$ in the usual manner (cf. Theorem 6.15). Show that

$$T^\dagger g(x) = \int_Y \overline{K(x, y)} g(y)\ d\nu(y) \text{ for all simple functions } g.$$
