Mathematics 245A
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Instructions: Try to do seven out of nine problems; they are all of equal value. There is plenty of working space, and a blank page at the end. Throughout this final, all measures are real (but could be signed or unsigned). You may use the axiom of choice freely.

You may enter in a nickname if you want your final score posted.

Good luck!

Name: ______________________

Nickame: ______________________

Student ID: ______________________

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Problem 1. ____________

Problem 2. ____________

Problem 3. ____________

Problem 4. ____________

Problem 5. ____________

Problem 6. ____________

Problem 7. ____________

Problem 8. ____________

Problem 9. ____________

Best 7 of 9: ______________________

Problem 1. Let X be a set, let $\mathcal{A}$ be a σ-algebra on X, and let μ and ν be two measures on $\mathcal{A}$. Suppose that μ is σ-finite and unsigned, and that ν is σ-finite and signed. Show that the following three statements are logically equivalent:

- (a) $-\mu \leq \nu \leq \mu$.
- (b) $|\nu| \leq \mu$.
- (c) $\nu \ll \mu$ and $|\frac{d\nu}{d\mu}(x)| \leq 1$ for μ-almost every $x \in X$.

There are six implications to prove here, but of course one can demonstrate as few as three of them to yield the result, and it would be smart to pick those three which are easiest to prove. Here we give all six implications.

(a) $\implies$ *(b)*: Let $X = X_- \cup X_+$ be a Jordan decomposition of X. If E is measurable in X, then

$$|\nu|(E) = |\nu|(E \cap X_-) + |\nu|(E \cap X_+) = \nu_-(E \cap X_-) + \nu_+(E \cap X_+) = -\nu(E \cap X_-) + \nu(E \cap X_+)$$

since ν_- is supported on X_- and ν_+ is supported on X_+. But since $\nu \leq \mu$ and $-\nu \leq \mu$, we have

$$-\nu(E \cap X_-) + \nu(E \cap X_+) \leq \mu(E \cap X_-) + \mu(E \cap X_+) \leq \mu(E)$$

and hence $|\nu| \leq \mu$. Note that the above arguments are still valid even if one of ν_+ or ν_- is infinite (recall that they cannot both be infinite).

(b) $\implies$ *(a)*: If E is measurable in X, then

$$\nu(E) = \nu_+(E) - \nu_-(E) \leq \nu_+(E) + \nu_-(E) = |\nu|(E) \leq \mu(E)$$

and

$$\nu(E) = \nu_+(E) - \nu_-(E) \geq -\nu_+(E) - \nu_-(E) = -|\nu|(E) \geq -\mu(E)$$

and hence $-\mu \leq \nu \leq \mu$.

(b) $\implies$ *(c)*: If E is measurable in X and $\mu(E) = 0$, then $|\nu(E)|$ must also be zero, which implies that $\nu_+(E) = \nu_-(E) = 0$, and hence $\nu(E) = 0$. Thus $\nu \ll \mu$, and $d\nu = \frac{d\nu}{d\mu} d\mu$. This implies $d|\nu| = |\frac{d\nu}{d\mu}| d\mu$. Pick any $\varepsilon > 0$, let A be any set of finite μ-measure, and let $E_\varepsilon := \{x \in A : |\frac{d\nu}{d\mu}| \geq 1 + \varepsilon\}$. Then $|\nu|(E_\varepsilon) \geq (1+\varepsilon)\mu(E_\varepsilon)$, but $\mu(E_\varepsilon)$ is finite and $|\nu(E_\varepsilon)| \leq \mu(E_\varepsilon)$, thus $\mu(E_\varepsilon) = 0$. Taking unions over countably many A (exploiting the σ-finite hypothesis) and over countably many ε going to zero, we see that the set where $|\frac{d\nu}{d\mu}| > 1$ has μ-measure zero as desired.

(c) $\implies$ *(b)*: Since $d|\nu| = |\frac{d\nu}{d\mu}| d\mu$ and $|\frac{d\nu}{d\mu}| \leq 1$ μ-a.e., we see that for any measurable E

$$|\nu|(E) = \int_E |\frac{d\nu}{d\mu}| d\mu \leq \int_E 1 d\mu = \mu(E).$$

$(a) \implies (c)$: If E is measurable in X and $\mu(E) = 0$, then $-\mu(E) \leq \nu(E) \leq \mu(E)$ and hence $\nu(E) = 0$. Thus $\nu \ll \mu$, and $d\nu = \frac{d\nu}{d\mu} d\mu$. Arguing as in $(b) \implies (c)$, we can then use the hypothesis $\nu \leq \mu$ to conclude that $\frac{d\nu}{d\mu} \leq 1$ μ-a.e., and using the hypothesis $\nu \geq -\mu$ we conclude that $\frac{d\nu}{d\mu} \geq -1$ μ-a.e. Combining these two, the claim follows.

$(c) \implies (a)$: For any measurable E,

$$\nu(E) = \int_E \frac{d\nu}{d\mu} d\mu \leq \int_E 1 d\mu = \mu(E)$$

and

$$\nu(E) = \int_E \frac{d\nu}{d\mu} d\mu \geq \int_E (-1) d\mu = -\mu(E).$$

Problem 2. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f_n \in L^1(X)$ be a sequence of absolutely integrable functions which converges in measure to another absolutely integrable function $f \in L^1(X)$. Let $g(x) := \sup_n |f_n(x)|$.

(a) Show that if g is absolutely integrable, then f_n converges to f in L^1.

Let f_{n_j} be an arbitrary subsequence of f_n. Then f_{n_j} also converges in measure to f, and thus there exists a subsequence $f_{n_{j_i}}$ which converges pointwise a.e. to f. By the dominated convergence theorem, this implies that $f_{n_{j_i}}$ converges in L^1 to f. Thus every subsequence of f_n itself contains a subsequence which converges to f in L^1, which forces the entire sequence to converge to L^1. (If f_n did not converge to f in L^1, there would be an ε such that $\|f_n - f\|_{L^1} > \varepsilon$ for infinitely many n, which would then produce a subsequence of f_n for which no convergent subsequence converged to f.)

One can also argue more directly without recourse to the dominated convergence theorem, but it requires some effort. One can reduce to the case of finite measures, because one can approximate g in L^1 by a function supported on a set of finite measure; for similar reasons one can reduce to the case when g is bounded. Then one can control the L^1 norm of $f_n - f$ without too much trouble from the convergence in measure hypothesis.

(b) Give an example to show that if g is not absolutely integrable, then f_n need not converge to f in L^1.

There are many examples which would work here; for instance take $X = \mathbf{R}$ with Lebesgue measure, and take $f_n = n 1_{[0,1/n]}$ and $f = 0$. Another somewhat different example is $f_n = \frac{1}{n} 1_{[0,n]}$ and $f = 0$. Note that one does not actually need to demonstrate that g is not absolutely integrable for these examples (though it is not hard to do), since this follows from the contrapositive of (a).

Problem 3. Let $(X, \mathcal{A}, \mu)$ be a *finite* measure space, and let f_n be a sequence of non-negative measurable functions which converge in the L^1 sense to zero.

(a) Show that $\sqrt{f_n}$ converges in the L^1 sense to zero. (Hint: for each n, split X into the region where $f_n \geq \varepsilon$ and where $f_n < \varepsilon$).

Pick any $\varepsilon > 0$, and we bound

$$\int_X |\sqrt{f_n}| \, d\mu \leq \int_{f_n \geq \varepsilon} \sqrt{f_n} \, d\mu + \int_{f_n < \varepsilon} \sqrt{f_n} \, d\mu.$$

If $f_n \geq \varepsilon$, then $\sqrt{f_n} \leq f_n / \sqrt{\varepsilon}$. On the other hand, if $f_n < \varepsilon$, then $\sqrt{f_n} \leq \sqrt{\varepsilon}$. Thus

$$\int_X |\sqrt{f_n}| \, d\mu \leq \int f_n / \sqrt{\varepsilon} \, d\mu + \int \sqrt{\varepsilon} \, d\mu.$$

Since $\int f_n$ converges to zero, we thus have

$$\limsup_{n \to \infty} \int_X |\sqrt{f_n}| \, d\mu \leq \int \sqrt{\varepsilon} \, d\mu = \sqrt{\varepsilon} \mu(X).$$

But $\varepsilon > 0$ is arbitrary and $\mu(X)$ is finite, and so we must have

$$\limsup_{n \to \infty} \int_X |\sqrt{f_n}| \, d\mu = 0$$

and the claim follows.

A slicker way to do it: use the arithmetic mean-geometric mean inequality to conclude that $\sqrt{f_n} \leq \frac{1}{2}(\varepsilon + f_n/\varepsilon)$, and one no longer needs to subdivide X. Even slicker: use the Cauchy-Schwarz inequality to conclude $\int_X \sqrt{f_n} \, d\mu \leq (\int_X f_n d\mu)^{1/2} (\int_X 1 d\mu)^{1/2}$.

(b) Give an example to show that the functions f_n^2 need not converge in the L^1 sense to zero.

Many examples here: one is to take $X = \mathbf{R}$ with Lebesgue measure and $f_n = n^{1/2} 1_{[0,1/n]}$. In fact one can find examples for which f_n^2 is not even absolutely integrable.

Problem 4. Let $\mathcal{B}$ be the Borel σ-algebra on the real line $\mathbf{R}$, and let μ be a finite non-negative Borel measure on $\mathcal{B}$. Let ν be counting measure on $\mathbf{R}$. Show that it is impossible for ν to be absolutely continuous with respect to μ, i.e. show that $\nu \not\ll \mu$.

Suppose for contradiction that $\nu \ll \mu$. Then all points $\{x\}$ on the real line have non-zero ν-measure, and hence must also have non-zero μ-measure. On the other hand, since μ is finite, we know that for each $\varepsilon > 0$ there are at most finitely many points x for which $\mu(\{x\}) > \varepsilon$; taking countable unions we conclude that $\mu(\{x\})$ can be non-zero for at most countably many points. Since the reals are uncountable, we obtain a contradiction.

Alternatively, one can use a divide-and-conquer algorithm to locate a point with zero μ measure. Start with an interval I, say the unit interval $[0, 1)$, and divide it into two subintervals. One of them will have at most half of the μ-measure of the whole. Pass to that interval, divide that into two subintervals, and then again locate the subinterval which has at most half of the μ measure of the previous interval (and thus at most one quarter of the whole). Repeating this process and using continuity from above we will locate a point in the real line with μ-measure zero but ν-measure one, thus contradicting absolute continuity.

Problem 5. Let X be a space, and let $X = A_1 \cup \ldots \cup A_n$ be a finite partition of X into n disjoint non-empty sets. Let $\mathcal{B}$ be the σ-algebra generated by $A_1, \ldots, A_n$ (i.e. the space of sets which are unions of some, none, or all of the A_i). Let μ be a finite unsigned measure on $\mathcal{B}$ such that $0 < \mu(A_i) < \infty$ for all $1 \leq i \leq n$, and let ν be a finite signed measure on $\mathcal{B}$. Show that ν is absolutely continuous with respect to μ, and that $\frac{d\nu}{d\mu}(x) = \frac{\nu(A_i)}{\mu(A_i)}$ for all $1 \leq i \leq n$ and all $x \in A_i$.

If E is measurable, then $E = \bigcup_{i \in I} A_i$ for some $I \subseteq \{1, \ldots, n\}$, and hence $\mu(E) = \sum_{i \in I} \mu(A_i)$. Thus $\mu(E) = 0$ can only occur if E is empty, which means that ν is automatically absolutely continuous with respect to μ. Now observe that

$$\nu(E) = \sum_{i \in I} \nu(A_i) = \sum_{i \in I} \frac{\nu(A_i)}{\mu(A_i)} \mu(A_i) = \int_E f \, d\mu$$

where $f(x) := \frac{\nu(A_i)}{\mu(A_i)}$ for all $1 \leq i \leq n$ and $x \in A_i$. Since E was an arbitrary measurable set, and since f is μ-measurable, we see from uniqueness of the Radon-Nikodym derivative that $\frac{d\nu}{d\mu} = f$ μ-a.e. But the only sets of μ-measure zero are the empty set, thus $\frac{d\nu}{d\mu} = f$ and the claim follows.

Problem 6. Let $F : \mathbf{R} \to \mathbf{R}$ be an increasing, right-continuous function, and let $\Phi : \mathbf{R} \to \mathbf{R}$ be a continuous increasing invertible function. Let μ_F and $\mu_{F\circ\Phi}$ be the Lebesgue-Stieltjes measures associated to F and $F \circ \Phi$ respectively. Show that if $f \in L^1(d\mu_F)$, then $f \circ \Phi \in L^1(d\mu_{F\circ\Phi})$, and

$$\int f \; d\mu_F = \int f \circ \Phi \; d\mu_{F\circ\Phi}.$$

It suffices to prove the claim for non-negative f, since the claim for general f then follows by taking positive and negative parts. It then suffices to prove the inequality

$$\int f \circ \Phi \; d\mu_{F\circ\Phi} \leq \int f \; d\mu_F$$

since by replacing Φ with Φ^{-1} we get the opposite inequality. Note that such an inequality will also yield for free that $f \circ \Phi \in L^1(d\mu_{F\circ\Phi})$ whenever $f \in L^1(d\mu_F)$. By monotone convergence (writing f as the monotone limit of simple functions) it suffices to verify this when f is a simple function; by linearity it suffices to verify this when $f = 1_E$ is an indicator function. Thus are reduced to checking that

$$\int 1_E \circ \Phi \; d\mu_{F\circ\Phi} \leq \int 1_E \; d\mu_F = \mu_F(E).$$

The right-hand side is the outer measure of E with respect to the premeasure of μ_F on half-open intervals. Thus if we cover E by countably many half-open intervals $I_1, I_2, \ldots$, it suffices to show that

$$\int 1_E \circ \Phi \; d\mu_{F\circ\Phi} \leq \sum_{n=1}^{\infty} \mu_F(I_n)$$

for any such covering. But by monotone convergence we have

$$\int 1_E \circ \Phi \; d\mu_{F\circ\Phi} \leq \int \sum_{n=1}^{\infty} 1_{I_n} \circ \Phi \; d\mu_{F\circ\Phi} = \sum_{n=1}^{\infty} \int 1_{I_n} \circ \Phi \; d\mu_{F\circ\Phi}$$

so it suffices to show that for each interval $I = [a, b)$, we have

$$\int 1_{[a,b)} \circ \Phi \; d\mu_{F\circ\Phi} \leq \mu_F([a, b)) = F(b) - F(a).$$

But this follows since $1_{[a,b)} \circ \Phi = 1_{[\Phi^{-1}(a), \Phi^{-1}(b))}$.

Problem 7. Let $(X, \mathcal{A}, \mu)$ be a measure space such that $\{x\} \in \mathcal{A}$ for all $x \in X$ (i.e. all points are measurable), and such that μ is a non-negative finite measure. Show that there is a unique decomposition $\mu = \mu_{pp} + \mu_c$, where μ_{pp} is a non-negative finite measure on $\mathcal{A}$ supported on a countable set (known as the "pure point" component of μ) and μ_c is a non-negative finite measure $\mathcal{A}$ such that $\mu_c(\{x\}) = 0$ for all $x \in X$ (known as the "continuous" component of μ). (Hint: Do *not* try to apply the Lebesgue-Radon-Nikodym theorem. A good place to start instead is to show that there are at most countably many points $x \in X$ for which $\mu(\{x\})$ is positive).

For any $\varepsilon > 0$, observe that the set of points x for which $\mu(\{x\}) > \varepsilon$ is at most finite, since μ itself is finite. Taking countable unions, we then conclude that the set $E := \{x : \mu(\{x\}) > 0\}$ is at most countable. Let $\mu_{pp} := \mu|_E$ be the restriction of μ to E, and let $\mu_c := \mu|_{E^c}$ be the restriction to the complement. Then μ_{pp} and μ_c are then also non-negative and finite, and μ_{pp} is supported on E and is thus supported on a countable set (one can enlarge E if necessary). Also if x is any point we have $\mu_c(\{x\}) = 0$ if $x \in E$ (since μ_c is restricted to E^c), and $\mu_c(\{x\}) = \mu(\{x\}) = 0$ if $x \in E^c$, by definition of E. This gives the existence of the decomposition $\mu = \mu_{pp} + \mu_c$.

Now suppose that there was an alternate decomposition $\mu = \mu'_{pp} + \mu'_c$. Then we have $\mu'_{pp} - \mu_p p = \mu_c - \mu'_c$ (note that all measures are finite). The left-hand side is supported on a countable set, but the right-hand side assigns a measure of zero to all points. Taking countable unions, we conclude that both sides are zero, and thus $\mu'_{pp} = \mu_{pp}$ and $\mu'_c = \mu_c$. This proves uniqueness.

Problem 8. Let $\mathcal{E}$ be the space of half-open rectangles $[a,b) \times [c,d)$ in the plane $\mathbf{R}^2$ (where $a < b$ and $c < d$ are real numbers). Let $\rho : \mathcal{E} \to \mathbf{R}^+$ be the function $\rho([a,b) \times [c,d)) := (b-a) \times (d-c)$, and let μ^* be the outer measure associated to ρ. Show that if E is a Lebesgue measurable subset of $\mathbf{R}^2$, then $\mu^*(E) = m(E)$, where m is the two-dimensional Lebesgue measure of E.

Let us first establish that $\mu^*(E) \geq m(E)$. Suppose we cover E by countably many half-open rectangles $R_1, R_2, \ldots$. Then

$$m(E) \leq \sum_{n=1}^{\infty} m(R_n) = \sum_{n=1}^{\infty} \rho(R_n)$$

since for each half-open rectangle it is clear that m and ρ co-incide. Taking infima over all coverings we conclude that $m(E) \leq \mu^*(E)$.

It remains to show that $\mu^*(E) \leq m(E)$. It suffices to do this when E is bounded, since an unbounded measurable set can be written as the countable union of bounded measurable sets and one can use countable additivity of m and countable subadditivity of μ^*.

Since $m(E) = m^*(E)$, it suffices to show that for any generalized rectangles $A_n \times B_n$ which cover E (where A_n, B_n are merely measurable subsets of $\mathbf{R}$), we have

$$\mu^*(E) \leq \sum_{n=1}^{\infty} m(A_n \times B_n).$$

Since E is bounded, we may take A_n and B_n to be bounded also. By subadditivity we have $\mu^*(E) \leq \sum_{n=1}^{\infty} \mu^*(A_n \times B_n)$, so it suffices to show that

$$\mu^*(A \times B) \leq m(A \times B) = m_1(A) m_1(B)$$

whenever A, B are bounded measurable subsets of $\mathbf{R}$ and m_1 denotes one-dimensional Lebesgue measure. But if we pick any ε, we can cover A by half-open intervals $I_1, \ldots$ such that $\sum_{i=1}^{\infty} m_1(I_i) \leq m_1^*(A) + \varepsilon = m_1(A) + \varepsilon$, and similarly we can find $J_1, \ldots$ covering B such that $\sum_{j=1}^{\infty} m_1(J_j) \leq m_1^*(B) + \varepsilon = m_1(B) + \varepsilon$. This implies that the rectangles $I_i \times J_j$ cover $A \times B$, thus

$$\mu^*(A \times B) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} m_1(I_i) m_1(J_j) = (\sum_{i=1}^{\infty} m_1(I_i))(\sum_{j=1}^{\infty} m_1(J_j)) \leq (m_1(A) + \varepsilon)(m_1(B) + \varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, and $m_1(A)$ and $m_1(B)$ are finite, the claim follows.

Problem 9. Let $\Phi : \mathbf{R}^n \to \mathbf{R}^n$ be a diffeomorphism (i.e. a continuously differentiable function whose inverse is also continuously differentiable), which is also a *contraction* in the sense that $|\Phi(x) - \Phi(y)| \leq |x - y|$ for all $x, y \in \mathbf{R}^n$ (where $|x - y|$ is the Euclidean distance between x and y). Show that $m(\Phi(E)) \leq m(E)$ for any Lebesgue measurable set E, where m denotes Lebesgue measure. (You may assume without proof that $\Phi(E)$ is Lebesgue measurable). Hint: it may help to first establish the case when E is a ball.

Using the change of variables theorem, we have

$$m(\Phi(E)) = \int 1_{\Phi(E)}\, dm = \int 1_E |\det(D\Phi)| dm$$

while

$$m(E) = \int 1_E dm.$$

Thus it will suffice to show that $|\det(D\Phi)(x)| \leq 1$ for all x. (Actually almost all x would suffice, but we can do all x here). There are two ways to do this:

Proof A. Suppose for contradiction that $|\det(D\Phi)(x)| > 1$ for some x; since Φ is continuously differentiable this implies that in fact $|\det(D\Phi)(x)| > 1 + \varepsilon$ for some ball $B(x, r)$ and some $\varepsilon > 0$. Using the change of variables theorem again, this implies that

$$m(\Phi(B(x, r))) \geq (1 + \varepsilon) m(B(x, r)).$$

But since Φ is a contraction, observe that if $y \in B(x, r)$ then $\Phi(y)$ is necessarily in $B(\Phi(x), r)$. Thus $\Phi(B(x, r)) \subseteq B(\Phi(x), r)$, and hence

$$m(\Phi(B(x, r))) \leq m(B(\Phi(x), r)) = m(B(x, r)),$$

a contradiction.

Proof B. Let $x \in \mathbf{R}^n$. Observe that $\det(D\Phi)$ is the volume of the parallelopiped with edge vectors $\frac{\partial \Phi}{\partial x_j}(x)$ for $j = 1, \ldots, n$. Thus it will suffice to show that each of these vectors has length at most 1. But this follows by using the definition

$$\frac{\partial \Phi}{\partial x_j}(x) = \lim_{h \to 0} \frac{\Phi(x + he_j) - \Phi(x)}{h}$$

and using the contraction property.