THE ABSOLUTE FUNCTIONAL CALCULUS FOR SECTORIAL OPERATORS

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by
TAMARA KUCHERENKO

Dr. Nigel Kalton, Dissertation Supervisor

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ABSTRACT

There are four chapters to this dissertation. The first chapter is based on joint work with Mark Hoffmann and Nigel Kalton. It is shown that R-bounded and weakly compact semigroups on L_1 and $C(K)$ can only exist for ℓ_1 and c_0. More generally, R-bounded weakly compact commuting approximating sequences in Banach spaces are discussed.

The second chapter is based on joint work with Lutz Weis and was produced while visiting the Universität Karlsruhe, Germany, on a DAAD research fellowship. As in chapter one, we focus on semigroups on L_1 spaces and the lack of H^∞-calculus thereof.

Precisely, we consider a sectorial operator A on a non-atomic L_p-space, $1 \leq p < \infty$, so that its resolvent consists of integral operators, or more generally, has a diffuse representation. Then the fractional domain spaces $\mathcal{D}(A^\alpha)$ for $\alpha \in (0,1)$ do not coincide with the real interpolation spaces of $(L_q, \mathcal{D}(A))$. As a consequence, we obtain that no such operator A has a bounded H^∞-calculus for $p = 1$.

The third chapter is demonstrating the pathological properties a sectorial operators on L_1 must have if it generates an R-bounded semigroup. Precisely we show that if A is R-sectorial and $\epsilon > 0$ then there is an invertible operator $U : L_1 \to L_1$ with $\|U - I\| < \epsilon$ such that for some positive Borel function w we have $U(\mathcal{D}(A)) \supset L_1(w)$. This actually improves some results from the first chapter. Roughly speaking, this means A is very close to a bounded operator. A central idea in the proof is to associate a family of representing measures to an R-bounded family of operators on L_1 and then show that it forms a weakly compact set.

The final chapter introduces absolute functional calculus for sectorial operators. We show that absolute calculus is stronger than H^∞-calculus and establish fundamental facts. We are able to improve a key theorem related to the maximal regularity problem and hence demonstrate the power and usefulness of our new concept. In trying to characterize spaces where sectorial operators have absolute calculus, we find that certain real interpolation spaces play a central role. We are then extending various known results in this setting. The idea of unifying theorems about sectorial operators on real interpolation spaces permeates our work and opens paths for future research on this subject.

TABLE OF CONTENTS

Chapter 1

R-bounded approximating sequences and applications to semigroups

1.1 Introduction

This chapter is based on joint work with M. Hoffmann and N. Kalton. Our results were published in [28].

It is shown that on certain Banach spaces, including $C[0,1]$ and $L_1[0,1]$, there is no strongly continuous semigroup $(T_t)_{0<t<1}$ consisting of weakly compact operators such that $(T_t)_{0<t<1}$ is an R-bounded family. More general results concerning approximating sequences are included and some variants of R-boundedness are also discussed.

Recent work on semigroup theory ([32], [53]) has highlighted the importance of the concept of R-boundedness . Let us recall the definition of R-bounded families of operators (cf. [6], [15], [11]).

Definition 1.1.1. A family $\mathcal{T}$ of operators in $\mathcal{L}(X,Y)$ is called *R-bounded* with R-boundedness constant $C > 0$ if letting $(\epsilon_k)_{k=1}^{\infty}$ be a sequence of independent Rademachers on some probability space then for every $x_1, \ldots, x_n \in X$ and $T_1, \ldots, T_n \in$

$\mathcal{T}$ we have

$$\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k T_k x_k\right\|^2\right)^{\frac{1}{2}} \leq C\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k x_k\right\|^2\right)^{\frac{1}{2}}. \tag{1.1}$$

By the Kahane-Khintchine inequality we can replace 2 above by any other exponent $1 \leq p < \infty$ to obtain an equivalent definition. We will also need the following definition introduced in [32].

Definition 1.1.2. A family $\mathcal{T}$ of operators in $\mathcal{L}(X,Y)$ is called *WR-bounded* with WR-boundedness constant $C > 0$ if for every $x_1, \ldots, x_n \in X, y_1^*, \ldots, y_n^* \in Y^*$ and $T_1, \ldots, T_n \in \mathcal{T}$ we have

$$\sum_{k=1}^{n} |\langle T_k x_k, y_k^*\rangle| \leq C\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k x_k\right\|^2\right)^{\frac{1}{2}}\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k y_k^*\right\|^2\right)^{\frac{1}{2}}. \tag{1.2}$$

It is clear by the Cauchy-Schwarz inequality that R-boundedness implies WR-boundedness. The converse is not true in general, but it holds for spaces with non-trivial type ([46], [32]).

In [32] it was shown that no reasonable differential operator on L_1 can have an $H^\infty-$calculus. In this note we consider the related question whether a differential-type operator on L_1 can generate an R-bounded semigroup. Note that if A is an R-sectorial operator (cf. [32]) with R-sectoriality angle less than $\frac{\pi}{2}$ then the semigroup $(e^{-tA})_{0<t<1}$ is necessarily R-bounded. In general, one expects a semigroup generated by a differential operator on a bounded domain to consist of weakly compact operators. We are thus led to consider the question whether one can have a strongly continuous semigroup $(T_t)_{0<t<1}$ on L_1 such that each T_t is weakly compact (or equivalently compact, since L_1 has the Dunford-Pettis property) and such that the family $(T_t)_{0<t<1}$ is R-bounded. In fact this leads to considering versions of the approximation property; the only property of the semigroup needed is commutativity. We consider the general question whether on a given separable Banach space one can find an R-bounded sequence $(T_n)_{n\in\mathbb{N}}$ of commuting weakly compact operators such that $\lim_{n\to\infty} T_n x = x$ for all $x \in X$. Our main results show that for the spaces $L_1[0,1]$, $C(K)$ (except c_0) and the disk algebra $A(\mathbb{D})$ this is impossible. These results may be regarded as extensions of classical results that the spaces $L_1, C(K)$ do not have unconditional bases [38].

In the case of L_1 we are led to consider a natural weakening of R-boundedness, where we use the definition (1.1) but only for single vectors.

Definition 1.1.3. A family $\mathcal{T}$ of operators in $\mathcal{L}(X,Y)$ is called *semi-R-bounded* if there is a constant $C > 0$ such that for every $x \in X, a_1, \dots, a_n \in \mathbb{C}$ and $T_1, \dots, T_n \in \mathcal{T}$ we have

$$\left(\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k a_k T_k x\right\|^2\right)^{\frac{1}{2}} \leq C \left(\sum_{k=1}^{n} |a_k|^2\right)^{\frac{1}{2}} \|x\|. \tag{1.3}$$

We note that semi-R-boundedness is equivalent to R-boundedness for operators on L_1. In Theorem 1.2.2 we actually characterize all spaces where semi-R-boundedness is equivalent to R-boundedness as spaces which are either Hilbert spaces or GT-spaces of cotype 2 in the terminology of Pisier [45].

The authors would like to thank Dale Alspach for drawing their attention to the notion of the ℓ_1 index of sequences.

1.2 R-boundedness and WR-boundedness

In this section, we make some remarks about R-boundedness and related notions.

Note that in a space of type 2, any uniformly bounded collection $\mathcal{T} \subset L(X, X)$ is semi-R-bounded. The converse is also true:

Proposition 1.2.1. *A Banach space X has type 2 if and only if uniform boundedness is equivalent to semi-R-boundedness.*

Proof. Suppose that every uniformly bounded family of operators is already semi-R-bounded. Pick any $x \in X$ and $x^* \in X^*$ such that $\|x\| = \|x^*\| = 1$ and $x^*(x) = 1$. Notice that the family $\mathcal{T} = \{x^* \otimes u : \|u\| = 1\}$ is uniformly bounded with constant one and hence semi-R-bounded by assumption. Let C be the semi-R-boundedness constant of $\mathcal{T}$. Select any $x_1, \ldots, x_n \in X$ and write $x_k = \|x_k\| u_k$ where $\|u_k\| = 1$. Then $\{x^* \otimes u_k : k = 1, \ldots, n\} \subset \mathcal{T}$ and

$$\begin{aligned}
\left(\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k x_k\right\|^2\right)^{\frac{1}{2}} &= \left(\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k \|x_k\| (x^* \otimes u_k) x\right\|^2\right)^{\frac{1}{2}} \\
&\leq C \left(\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k \|x_k\| x\right\|^2\right)^{\frac{1}{2}} \\
&= C\|x\| \left(\sum_{k=1}^{n} \|x_k\|^2\right)^{\frac{1}{2}} \\
&= C \left(\sum_{k=1}^{n} \|x_k\|^2\right)^{\frac{1}{2}}
\end{aligned}$$

Thus, X has type 2. □

For some spaces, semi-R-boundedness is equivalent to R-boundedness and we are able to completely characterize these spaces in the next theorem. Let us recall that a Banach space X is called a *GT-space* if every bounded operator $T : X \to \ell_2$

is absolutely summing. Examples of GT-spaces of cotype 2 are L_1, the quotient of L_1 by a reflexive subspace [45],[33], and L_1/H_1 [13]. It is unknown whether every GT-space has cotype 2.

Theorem 1.2.2. *Suppose X is separable. Then the following are equivalent :*

(i) Every semi-R-bounded family of operators on X is R-bounded.

(ii) X is isomorphic to ℓ_2 or X is a GT-space of cotype 2.

Proof. First we prove that (i) implies (ii). Suppose that every semi-R-bounded family of operators on X is R-bounded. Let us note that this implies the existence of a constant K so that if $\mathcal{T}$ has semi-R-boundedness constant C then it has R-boundedness constant KC; for otherwise we could find a sequence $\mathcal{T}_n$ of families with semi-R-boundedness constant 1 and R-boundedness constant at least 4^n; then the family $\cup_{n\geq 1} 2^{-n}\mathcal{T}_n$ contradicts our assumption. Fix $M > 1$ and take $x \in X$. Choose $n \in \mathbb{N}$. By Dvoretzky's theorem [40] we can find $e_1, \ldots, e_n \in X$ such that for any $a_1, \ldots, a_n \in \mathbb{C}$ we have

$$M^{-1}\left(\sum_{k=1}^{n} |a_k|^2\right)^{\frac{1}{2}} \leq \left\|\sum_{k=1}^{n} a_k e_k\right\| \leq M\left(\sum_{k=1}^{n} |a_k|^2\right)^{\frac{1}{2}}.$$

Consider the family of operators $\mathcal{T}_n = \{u^* \otimes e_k : \|u^*\| = 1, k = 1, \ldots, n\}$. Then each $\mathcal{T}_n$ is semi-R-bounded with constant M as follows. A finite subfamily of $\mathcal{T}_n$ is of the form $\{u^*_{kj} \oplus e_k : 1 \leq k \leq n, 1 \leq j \leq m_k\}$ for some $m_1, \ldots, m_n \in \mathbb{N}$. Then for every $a_{11}, \ldots, a_{nm_n} \in \mathbb{C}$ we have, (letting ϵ_{kj} denote independent Rademachers),

$$\begin{aligned}
&\left(\mathbb{E}\left\|\sum_{k=1}^{n}\sum_{j=1}^{m_k}\epsilon_{kj}a_{kj}u_{kj}^*(x)e_k\right\|^2\right)^{\frac{1}{2}} \\
&\leq M\left(\mathbb{E}\sum_{k=1}^{n}\left|\sum_{j=1}^{m_k}\epsilon_{kj}u_{kj}^*(x)a_{kj}\right|^2\right)^{\frac{1}{2}} \\
&\leq M\left(\sum_{k=1}^{n}\mathbb{E}\left|\sum_{j=1}^{m_k}\epsilon_{kj}u_{kj}^*(x)a_{kj}\right|^2\right)^{\frac{1}{2}} \\
&\leq M\left(\sum_{k=1}^{n}\sum_{j=1}^{m_k}|a_{kj}|^2\right)^{\frac{1}{2}}\|x\|
\end{aligned}$$

Our assumption implies that each $\mathcal{T}_n$ is R-bounded with constant KM. Let $x_1, \ldots, x_n \in X$ and write $x_k = \|x_k\|u_k$ where $\|u_k\| = 1$. Choose $u_k^* \in X^*$ such that $u_k^*(u_k) = 1$ and $\|u_k^*\| = 1$. Now we have

$$\begin{aligned}
\left(\sum_{k=1}^{n}\|x_k\|^2\right)^{\frac{1}{2}} &\leq M\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k\|x_k\|e_k\right\|^2\right)^{\frac{1}{2}} \\
&= M\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k\|x_k\|u_k^*(u_k)e_k\right\|^2\right)^{\frac{1}{2}} \\
&= M\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k\|x_k\|(u_k^*\otimes e_k)(u_k)\right\|^2\right)^{\frac{1}{2}} \\
&\leq KM^2\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k\|x_k\|u_k\right\|^2\right)^{\frac{1}{2}} \\
&= KM^2\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k x_k\right\|^2\right)^{\frac{1}{2}}
\end{aligned}$$

This shows that X has cotype 2.

Let us assume that X has non-trivial type. Then by results of Pisier [45] and also by Figiel and Tomczak-Jaegermann [25], ℓ_2^n is uniformly complemented in X.

Thus, for some constant C, for every $n \in \mathbb{N}$ we can choose a biorthogonal system $\{(e_k, e_k^*) : k = 1, \dots, n\}$ in $X \times X^*$ such that

$$\left\| \sum_{k=1}^{n} a_k e_k \right\| \leq C \left(\sum_{k=1}^{n} |a_k|^2 \right)^{\frac{1}{2}}$$

and

$$\left\| \sum_{k=1}^{n} a_k e_k^* \right\| \leq C \left(\sum_{k=1}^{n} |a_k|^2 \right)^{\frac{1}{2}}$$

for all $a_1, \dots, a_n \in \mathbb{C}$. Note that for any $x \in X$ and $a_1, \dots, a_n \in \mathbb{C}$,

$$\left| \sum_{k=1}^{n} a_k e_k^*(x) \right| \leq C \left(\sum_{k=1}^{n} |a_k|^2 \right)^{\frac{1}{2}} \|x\|$$

and so

$$\left(\sum_{k=1}^{n} |e_k^*(x)|^2 \right)^{\frac{1}{2}} \leq C\|x\|.$$

Consider the family of operators $\mathcal{T}_n = \{e_k^* \otimes u : \|u\| = 1; k = 1, \dots, n\}$. Let $x \in X$. Then for any $a_1, \dots, a_n \in \mathbb{C}$ and every $u_1, \dots, u_n \in X$ of norm one we have

$$\begin{aligned}
\mathbb{E} \left\| \sum_{k=1}^{n} \epsilon_k a_k (e_k^* \otimes u_k)(x) \right\| &= \mathbb{E} \left\| \sum_{k=1}^{n} \epsilon_k a_k e_k^*(x) u_k \right\| \\
&\leq \sum_{k=1}^{n} \|a_k e_k^*(x) u_k\| \\
&= \sum_{k=1}^{n} |a_k||e_k^*(x)| \\
&\leq \left(\sum_{k=1}^{n} |a_k|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} |e_k^*(x)|^2 \right)^{\frac{1}{2}} \\
&\leq C \left(\sum_{k=1}^{n} |a_k|^2 \right)^{\frac{1}{2}} \|x\|.
\end{aligned}$$

We conclude that $\mathcal{T}_n$ is semi-R-bounded with constant C and hence $\mathcal{T}_n$ is R-bounded for constant KC independent of n. This implies that X has type 2 as follows. Choose any $x_1, \dots, x_n \in X$ and write $x_k = \|x_k\| u_k$ where $\|u_k\| = 1$. Then

$$
\begin{aligned}
\left(\mathbb{E} \left\| \sum_{k=1}^{n} \epsilon_k x_k \right\|^2 \right)^{\frac{1}{2}} &= \left(\mathbb{E} \left\| \sum_{k=1}^{n} \epsilon_k \|x_k\| u_k \right\|^2 \right)^{\frac{1}{2}} \\
&= \left(\mathbb{E} \left\| \sum_{k=1}^{n} \epsilon_k \|x_k\| e_k^*(e_k) u_k \right\|^2 \right)^{\frac{1}{2}} \\
&= \left(\mathbb{E} \left\| \sum_{k=1}^{n} \epsilon_k \|x_k\| (e_k^* \otimes u_k)(e_k) \right\|^2 \right)^{\frac{1}{2}} \qquad (1.4) \\
&\leq KC \left(\mathbb{E} \left\| \sum_{k=1}^{n} \epsilon_k \|x_k\| e_k \right\|^2 \right)^{\frac{1}{2}} \\
&\leq KC^2 \left(\sum_{k=1}^{n} \|x_k\|^2 \right)^{\frac{1}{2}}
\end{aligned}
$$

Now, X has type 2 and cotype 2 and is therefore isomorphic to ℓ_2 by Kwapien's theorem [56].

Now suppose on the contrary that X has trivial type. We will show that X is a GT-space, i.e. any $T : X \longrightarrow \ell_2$ is 1-summing. Fix $T : X \longrightarrow \ell_2$ of norm one. Since X has cotype 2 we can equivalently show that any such T is 2-summing [19]. It suffices to check that for any $n \in \mathbb{N}$ and operator $S : \ell_2^n \longrightarrow X$ such that $\|S\| \leq 1$ we have $\pi_2(TS) \leq C$ where C does not depend on n [56]. One can assume that $TS : \ell_2^n \longrightarrow \ell_2^n$ and that TS is diagonal with respect to the canonical orthonormal basis (e_k) in ℓ_2^n, i.e. $TSe_k = \lambda_k e_k$ for some $\lambda_1, \dots, \lambda_n$. Then it suffices to show uniform boundedness of the Hilbert-Schmidt norms $\|TS\|_{HS} = (\sum_{k=1}^{n} \|TSe_k\|^2)^{\frac{1}{2}}$. Write $f_k^* = T^* e_k^* \in X^*$ and $f_k = Se_k \in X$. Consider $\{f_k^* \otimes u : k = 1, \dots, n; \|u\| = 1\}$. We will show that this family is semi-R-bounded with constant one. Take

$u_1, \ldots, u_n \in X$ of norm one and $a_1, \ldots, a_n \in \mathbb{C}$. Then for $x \in X$ we have

$$\begin{aligned}
\mathbb{E}\left\| \sum_{k=1}^{n} \epsilon_k a_k f_k^*(x) u_k \right\| &\leq \sum_{k=1}^{n} |a_k| |f_k^*(x)| \\
&\leq \left(\sum_{k=1}^{n} |a_k|^2 \right)^{\frac{1}{2}} \left(\sum_{k=1}^{n} |e_k^*(Tx)|^2 \right)^{\frac{1}{2}} \\
&= \left(\sum_{k=1}^{n} |a_k|^2 \right)^{\frac{1}{2}} \|Tx\| \\
&\leq \left(\sum_{k=1}^{n} |a_k|^2 \right)^{\frac{1}{2}} \|x\|
\end{aligned}$$

Therefore, $\{f_k^* \otimes u : k = 1, \ldots, n; \|u\| = 1\}$ is R-bounded with constant K.

Since X has trivial type, it contains ℓ_1^n uniformly [45]. Hence, for fixed $M > 1$ and every $n \in \mathbb{N}$ there are $y_1, \ldots, y_n \in X$ with $\|y_k\| = 1$ for $1 \leq k \leq n$ such that

$$\sum_{k=1}^{n} |a_k| \leq M \left\| \sum_{k=1}^{n} a_k y_k \right\| \tag{1.5}$$

Choose any scalars $b_1, \ldots, b_n$. Now using R-boundedness and Kahane's inequality for $p = 1$ with constant A we have

$$\begin{aligned}\sum_{k=1}^{n}|b_k||\lambda_k| &= \sum_{k=1}^{n}|b_k||f_k^*(f_k)| \\ &\le M\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k b_k f_k^*(f_k)y_k\right\|^2\right)^{\frac{1}{2}} \\ &= M\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k b_k (f_k^*\otimes y_k)(f_k)\right\|^2\right)^{\frac{1}{2}} \\ &\le KM\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k b_k f_k\right\|^2\right)^{\frac{1}{2}} \\ &\le KM\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k b_k e_k\right\|^2\right)^{\frac{1}{2}} \\ &\le KM\left(\sum_{k=1}^{n}|b_k|^2\right)^{\frac{1}{2}}\end{aligned}$$

Thus,

$$\left(\sum_{k=1}^{n}|\lambda_k|^2\right)^{\frac{1}{2}} \le KM$$

and so $\|TS\|_{HS} \le KM$. Therefore, any operator $T : X \longrightarrow \ell_2$ is 2-summing. This completes the proof of (i) implies (ii).

Now we will show that (ii) implies (i). Suppose that X is a GT-space of cotype 2, and that $\mathcal{T}$ is a family of semi-R-bounded operators. We will show that $\mathcal{T}$ is R-bounded. Since X is separable, there is a quotient map $Q : \ell_1 \longrightarrow X$. First, we show that any semi-R-bounded family of operators from ℓ_1 into X is already R-bounded. Let $\mathcal{S}$ be such a family with semi-R-boundedness constant one. Suppose $S_1, \ldots, S_n \in \mathcal{S}$ and $x_1, \ldots, x_n \in \ell_1$. Then $x_k = \sum_{j=1}^{\infty}\xi_{jk}e_j$ where (e_j) is the canonical basis of ℓ_1.

Let us denote by C the constant in the Kahane-Khintchine inequality for any

Banach space:

$$\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k x_k\right\|^2\right)^{\frac{1}{2}} \le C\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k x_k\right\|.$$

Thus

$$\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k S_k x_k\right\|^2\right)^{\frac{1}{2}} \le C\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k S_k x_k\right\|$$

Then

$$\begin{aligned}
\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k S_k x_k\right\| &= \mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k S_k\sum_{j=1}^{\infty}\xi_{jk}e_j\right\| \\
&\le \sum_{j=1}^{\infty}\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k\xi_{jk}S_k e_j\right\| \\
&\le \sum_{j=1}^{\infty}\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k\xi_{jk}S_k e_j\right\|^2\right)^{\frac{1}{2}} \\
&\le \sum_{j=1}^{\infty}\left(\sum_{k=1}^{n}|\xi_{jk}|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Combining and using the Khintchine inequality again we obtain

$$\begin{aligned}
\left(\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k S_k x_k\right\|^2\right)^{\frac{1}{2}} &\le C^2\sum_{j=1}^{\infty}\mathbb{E}\left|\sum_{k=1}^{n}\epsilon_k\xi_{jk}\right| \\
&= C^2\mathbb{E}\left(\sum_{j=1}^{\infty}\left|\sum_{k=1}^{n}\epsilon_k\xi_{jk}\right|\right) \\
&= C^2\mathbb{E}\left\|\sum_{j=1}^{\infty}\sum_{k=1}^{n}\epsilon_k\xi_{jk}e_j\right\|_{\ell_1} \\
&= C^2\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k\sum_{j=1}^{\infty}\xi_{jk}e_j\right\| \\
&= C^2\mathbb{E}\left\|\sum_{k=1}^{n}\epsilon_k x_k\right\|
\end{aligned}$$

Combining the previous two computations gives that $\mathcal{S}$ is R-bounded.

Now let $\mathcal{T}$ be a family of operators on X with semi-boundedness constant one. Let $Q : \ell_1 \to X$ be a quotient map and note that the family $\mathcal{S} = \{TQ : T \in \mathcal{T}\}$ is R-bounded with some constant B by the above calculation.

We will apply a characterization of GT-spaces of cotype 2 due to Pisier [45].

Proposition 1.2.3 (Pisier). *X is a GT-space of cotype 2 if and only if there is a constant $C > 0$ such that for any $n \in \mathbb{N}, x_1, \ldots, x_n \in X$ there are $y_1, \ldots, y_n \in \ell_1$ such that $Qy_k = x_k, k = 1, \ldots, n$ and*

$$\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k y_k\right\| \leq C\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k x_k\right\| \tag{1.6}$$

Now take $n \in \mathbb{N}, T_1, \ldots, T_n \in \mathcal{T}$ and $x_1, \ldots, x_n \in X$. Choose $y_1, \ldots, y_n \in \ell_1$ according to 1.2.3. Then

$$\begin{aligned}
\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k T_k x_k\right\| &= \mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k T_k Q y_k\right\| \\
&\leq B\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k y_k\right\| \\
&\leq CB\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k x_k\right\|
\end{aligned}$$

Thus, $\mathcal{T}$ is R-bounded. The proof is complete. □

For a set $\mathcal{T}$ of bounded linear operators we will use the notation $\mathcal{T}^* = \{T^* : T \in \mathcal{T}\}$.

Lemma 1.2.4. *(i) If $\mathcal{T}$ is R-bounded then $\mathcal{T}^{**}$ is R-bounded (with the same constant).*

(ii) If $\mathcal{T}$ is WR-bounded then $\mathcal{T}^*$ and $\mathcal{T}^{**}$ are WR-bounded (with the same constant).

(iii) If $\mathcal{T}$ is semi-R-bounded then $\mathcal{T}^{**}$ is semi-R-bounded (with the same constant).

Proof. The proofs of (i) and (iii) are similar. For (i) suppose $T_1, \dots, T_n \in \mathcal{T}$ and that $\mathcal{T}$ has R-boundedness constant one. Let $\Omega = \{-1, 1\}^n$ with $\mathbb{P}$ normalized counting measure on Ω. Let ϵ_k be the sequence of coordinate maps on Ω. Let $\mathrm{Rad}\,(\Omega; X)$ be the subspace of $L_2(\Omega, \mathbb{P}; X)$ generated by the functions $\epsilon_k \otimes x$ for $1 \le k \le n$ and $x \in X$ (this space is isomorphic to X^n). Then $\mathrm{Rad}\,(\Omega; X^{**})$ can be identified naturally with a subspace of $\mathrm{Rad}\,(\Omega; X)^{**}$. Consider the map $\mathbf{T} : \mathrm{Rad}\,(\Omega; X) \to \mathrm{Rad}\,(\Omega; X)$ defined by

$$\mathbf{T}\left(\sum_{k=1}^{n} \epsilon_k \otimes x_k\right) = \sum_{k=1}^{n} \epsilon_k \otimes T_k x_k.$$

Then $\|\mathbf{T}\| \le 1$ and so $\|\mathbf{T}^{**}\| \le 1$ and (i) follows.

Let us now prove (ii). Suppose $\mathcal{T}$ is WR-bounded with constant one and $T_1, \dots, T_n \in \mathcal{T}$. Suppose $x_1^*, \dots, x_n^* \in X^*$ are such that

$$\left(\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k x_k^*\right\|^2\right)^{\frac{1}{2}} \le 1.$$

Then, using the identification of $\mathrm{Rad}\,(\Omega, X^{**})$ as the bidual of $\mathrm{Rad}\,(\Omega, X)$ we observe that the set of functions of the form $\sum_{k=1}^{n} \epsilon_k x_k^{**}$ in $\mathrm{Rad}\,(\Omega, X^{**})$ such that

$$\sum_{k=1}^{n} |\langle T_k^* x_k^*, x_k^{**}\rangle| \le 1$$

is weak*-closed and contains the unit ball of $\mathrm{Rad}\,(\Omega, X)$. By Goldstine's theorem it contains the unit ball of $\mathrm{Rad}\,(\Omega, X^{**})$ and this implies that $\mathcal{T}^*$ is WR-bounded with constant one. $\square$

Now it is time to give an example of a family of operators that is uniformly bounded but not WR-bounded. The previous lemma will imply that the corresponding dual family is semi-R-bounded but not WR-bounded.

Example. Let $X = \ell_p, 1 \leq p < 2$. Pick any non-zero element $x \in X$ and choose $u^* \in X^*$ of norm one such that $u^*(x) \neq 0$. Define $T_k = u^* \otimes e_k$ where (e_k) is the canonical basis of X. The family $\{T_k\}$ is uniformly bounded, $\|T_k\| = 1$, but we will show that it is not wR-bounded. Consider the dual basis (e_k^*) in $(\ell_p)^*$. Then

$$\sum_{k=1}^{n} |\langle T_k x, e_k^* \rangle| = \sum_{k=1}^{n} |\langle u^*(x) e_k, e_k^* \rangle| = n|u^*(x)| \tag{1.7}$$

On the other hand, we have

$$\left(\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k x\right\|^2\right)^{\frac{1}{2}} \left(\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k e_k^*\right\|^2\right)^{\frac{1}{2}} = \|x\| n^{\frac{1}{2}} n^{1/q} \tag{1.8}$$

Here q satisfies $1/p + 1/q = 1$. If $p < 2$ then $q > 2$ and $\frac{1}{2} + 1/q < 1$, so for $1 \leq p < 2$ the family $\{T_k\}$ can not be wR-bounded.

We have $T_k^* = e_k^{**} \otimes u^*$ on $X^* = \ell_q$ where $2 < q \leq \infty$. Consider $q \neq \infty$. Since by reflexivity $T_k^{**} = T_k$ and using 1.2.4 we see that $\{T_k^*\}$ is not WR-bounded. However, X^* has type 2 and hence $\{T_k^*\}$ is semi-R-bounded by 1.2.1.

1.3 The main results

Suppose X is any Banach space. We shall say that a sequence $\mathcal{T} = (T_k)_{k=1}^{\infty}$ is an *approximating sequence* if $\lim_{k \to \infty} \|x - T_k x\| = 0$ for every $x \in X$. We will say that $\mathcal{T}$ is compact (relatively, weakly compact) if each T_k is compact (relatively, weakly compact). We will say that $\mathcal{T}$ is commuting if we have $T_k T_l = T_l T_k$ for $l, k \in \mathbb{N}$.

If $\mathcal{T}$ is a commuting approximating sequence, let us define the subspace $E_{\mathcal{T}}$ of X^* to be the closed linear span of $\cup_k T_k^*(X^*)$. The following Lemma is trivial.

Lemma 1.3.1. *If $\mathcal{T}$ is a commuting approximating sequence then $E_{\mathcal{T}}$ is a norming subspace of X^*, i.e. for some C we have*

$$\|x\| \leq C \sup_{x^* \in B_{E_{\mathcal{T}}}} |x^*(x)| \qquad x \in X$$

and $\mathcal{T}$ is weakly compact, $\lim_{n\to\infty} T_n^* x^* = x^*$ weakly for $x^* \in \mathcal{E}_{\mathcal{T}}$.

$(T_n^*|_{E_{\mathcal{T}}})_{n=1}^{\infty}$ is an approximating sequence for $E_{\mathcal{T}}$.

Let us recall that a Banach space X has property (V) of Pełczyński if every unconditionally converging operator $T : X \to Y$ is weakly compact. The spaces $C(K)$ have property (V) [42] and more generally any C^*-algebra has property (V) [44]. The disk algebra $A(\mathbb{D})$ also has property (V) [33], [17]; see also [49]. We also recall that a Banach space X is said to have property (V^*) if whenever (x_n) is a bounded sequence in X then either:

(i) (x_n) has a subsequence which is weakly Cauchy, or

(ii) (x_n) has a subsequence (y_n) such that for some sequence (y_n^*) in X^* and $\delta > 0$ we have $|y_n^*(y_n)| \geq \delta$ and

$$\left\| \sum_{k=1}^{n} a_k y_k^* \right\| \leq \max_{1\leq k\leq n} |a_k| \qquad a_1, \ldots, a_n \in \mathbb{C},\ n \in \mathbb{N}.$$

Property (V^*) was introduced by Pełczyński [42]. We note that Bombal [8] shows that every Banach lattice not containing c_0 has property (V^*). Any subspace of a space with property (V^*) also has property (V^*).

Lemma 1.3.2. *Let X, Y be Banach spaces and let $\mathcal{T} = (T_k)_{k=1}^{\infty}$ be any sequence of operators in $\mathcal{L}(X, Y)$. Suppose either (i) $\mathcal{T}$ is semi-R-bounded or (ii) $\mathcal{T}$ is WR-bounded and Y has property (V^*). Then for every $x \in X$ the sequence $(T_k x)_{k=1}^{\infty}$ has a weakly Cauchy subsequence.*

Proof. If not, by passing to a subsequence we can suppose $(T_k x)_{k=1}^{\infty}$ is equivalent to the canonical ℓ_1-basis ([48], [41]). If $\mathcal{T}$ is semi-R-bounded we observe that for some C we have

$$\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k a_k T_k x\right\| \leq C\left(\sum_{k=1}^{n} |a_k|^2\right)^{\frac{1}{2}} \|x\|, \qquad a_1, \ldots, a_n \in \mathbb{C},\ n \in \mathbb{N}.$$

This gives a contradiction.

In case (ii), we can pass to a subsequence and assume the existence of $y_n^* \in Y^*$ such that

$$\left\|\sum_{k=1}^{n} a_k y_k^*\right\| \leq \max_{1\leq k\leq n} |a_k| \qquad a_1, \ldots, a_n \in \mathbb{C},\ n \in \mathbb{N}$$

and $|y_n^*(Tx_n)| \geq \delta > 0$ for all n. Then

$$\begin{aligned} n\delta &\leq \sum_{k=1}^{n} |y_k^*(T_k x)| \\ &\leq C\left(\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k x\right\|^2\right)^{\frac{1}{2}} \left(\mathbb{E}\left\|\sum_{k=1}^{n} \epsilon_k y_k^*\right\|^2\right)^{\frac{1}{2}} \\ &\leq C\sqrt{n}. \end{aligned}$$

This also yields a contradiction. □

Theorem 1.3.3. *Let X be a Banach space with a commuting weakly compact approximating sequence $\mathcal{T}$. Suppose either that (i) $\mathcal{T}$ is semi-R-bounded and X is weakly sequentially complete or (ii) $\mathcal{T}$ is WR-bounded and X has property (V^*). Then X is isomorphic to a dual space.*

Proof. In either case we consider the family $\mathcal{T}^{**} \subset \mathcal{L}(X^{**}, X)$. By Lemma 4.2 for each $x^{**} \in X^{**}$ we can find a subsequence $T_{k_n}^{**}x^{**}$ so that $T_{k_n}^{**}(x^{**})$ is weakly convergent to some $y \in X$. Then for $x^* \in X^*$

$$x^*(T_k y) = \lim_{n\to\infty} x^*(T_k T_n^{**} x^{**}) = \lim_{n\to\infty} x^*(T_n T_k^{**} x^{**})$$

so that $T_k y = T_k^{**}x^{**}$. Hence $\lim_{k\to\infty} \|y - T_k^{**}x^{**}\| = 0$.

We now show that $E_{\mathcal{T}}^*$ can be identified with X. Clearly X canonically embeds in $E_{\mathcal{T}}^*$ since $E_{\mathcal{T}}$ is norming. If $f^* \in E_{\mathcal{T}}^*$ then by the Hahn-Banach theorem there exists $x^{**} \in X^{**}$ with $\|x^{**}\| = \|f^*\|$ and $x^{**}(x^*) = f^*(x^*)$ for $x^* \in E_{\mathcal{T}}$. Let $y = \lim_{k\to\infty} T_k^{**}x^{**}$. Then for $x^* \in E_{\mathcal{T}}$,

$$x^*(y) = \lim_{k\to\infty} x^*(T_k^{**}x^{**}) = \lim_{k\to\infty} x^{**}(T_k^* x^*) = f^*(x^*).$$

Hence $E_{\mathcal{T}}^* = X$. □

Theorem 1.3.4. *The space $L_1(0,1)$ does not have a commuting weakly compact approximating sequence which is either semi-R-bounded or WR-bounded.*

Proof. L_1 is not a dual space [56]. □

Of course a semi-R-bounded sequence in L_1 is actually R-bounded.

Theorem 1.3.5. *Let X be a separable Banach space with property (V). If X has a commuting weakly compact approximating sequence $(T_n)_{n=1}^\infty$ which is WR-bounded, then X^* is separable and has a WR-bounded commuting weakly compact approximating sequence.*

Proof. Since X has (V), it follows that X^* has property (V^*). We show that $\lim_{n\to\infty} T_n^* x* = x^*$ weakly for $x^* \in X^*$. Indeed $T_n^* x^*$ converges weak* to x^* and it must have a weakly convergent subsequence by Lemma 4.2. Hence $x^* \in E_{\mathcal{T}}$ so $X^* = E_{\mathcal{T}}$. Now $T_n^*(B_{X^*})$ is weakly compact by Gantmacher's theorem also and weak*-metrizable, hence norm separable. Thus X^* is separable, and so by Mazur's theorem and a diagonal argument we can find a sequence of convex combinations $(S_n^*)_{n=1}^{\infty}$ of $(T_n^*)_{n=1}^{\infty}$ which is an approximating sequence. □

Corollary 1.3.6. *If K is an uncountable compact metric space then $C(K)$ has no WR-bounded commuting weakly compact approximating sequence. The disk algebra has no WR-bounded weakly compact approximating sequence.*

We now consider $C(K)$ when K is countable. In this case $C(K)$ is homeomorphic to a space $C(\alpha) = C([1,\alpha])$ where α is a countable ordinal. There is a characterization of such $C(K)$ due to Bessaga and Pełczyński [7].

Theorem 1.3.7 (Bessaga-Pełczyński). *If $\alpha < \beta, C(\omega^\alpha \cdot k)$ is isomorphic to $C(\omega^\beta \cdot n)$ if and only if $\beta < \alpha \cdot \omega$. Consequently, $C(\omega^{\omega^\gamma})$, $0 \leq \gamma < \omega_1$, is a complete list of representatives of the isomorphism classes of $C(K)$ for K a countable compact metric space.*

The following lemma can be obtained as an applications of ℓ_1-indices ([1], [9], [10]). However, for convenience of the reader we will give a direct proof by construction.

Lemma 1.3.8. *Let α be a countable ordinal with $\alpha \geq \omega^\omega$. Then there exists $f \in C(\alpha)^{**}$ so that whenever $f_n \in C(\alpha)$ converges to $f \in C(\alpha)^{**}$ weak* then for any $m \in \mathbb{N}$ there exist $n_1, \ldots, n_m \in \mathbb{N}$ such that*

$$\left\| \sum_{k=1}^{m} \epsilon_k f_{n_k} \right\| \geq \tfrac{1}{2} m \qquad \epsilon_k = \pm 1, \quad k = 1, 2, \ldots, m.$$

Proof. In this case $C(\alpha)^{**}$ can be identified with $\ell_\infty(\alpha)$. It is easy to see that it suffices to consider the case $\alpha = \omega^\omega$.

Consider $f \in X^{**}$ defined by $f(\sum_{k=0}^N \omega^k l_k) = (-1)^{\sum_{k=0}^N l_k}$ and $f(\omega^\omega) = 1$. Writing K for the space $[1, \omega^\omega]$ let $K^{(p)}$ denote the $p-$th derived set of K. Then $K^{(p)}$ consists of all ordinals of the form $\sum_{k=p}^n \omega^k l_k$ together with ω^ω. For each $p \in \mathbb{N}$, $K^{(p)}$ is nonempty. Furthermore for each $\alpha \in K^{(p)}$ and every open neighborhood V of α we have that f takes both values ± 1 on $V \cap K^{(p-1)}$.

Let $f_n \in C(K)$ be any sequence such that (f_n) converges to f weak*.

Fix $0 < \delta < \frac{1}{2}$ and $m \in \mathbb{N}$. We construct $(f_{n_1}, \ldots, f_{n_m})$ inductively. We start from $K^{(m)}$. By definition of f we can pick $\alpha_1^1, \alpha_2^1 \in K^{(m)}$ such that $f(\alpha_j^1) = (-1)^j$ for $j = 1, 2$. Then find $n_1 \in \mathbb{N}$ such that $|f_{n_1}(\alpha_j^1) - (-1)^j| < \delta$. Since f_{n_1} is continuous we can choose open neighborhoods U_j^1 of α_j^1 such that $|f_{n_1}(\alpha) - (-1)^j| < \delta$ for all $\alpha \in U_j^1$.

For the inductive step, suppose that $(n_j)_{j=1}^k$, $(\alpha_j^k)_{j=1}^{2^k}$ and open sets $(U_j^k)_{j=1}^{2^k}$ have been chosen so that $\alpha_j^k \in U_j^k$. Then for $i = 1, \ldots, 2^k$ find points $\alpha_{2i-1}^{k+1}, \alpha_{2i}^{k+1} \in U_i^k \cap K^{(m-k+1)}$ with $f(\alpha_j^{k+1}) = (-1)^j$. By pointwise convergence, we can select $n_{k+1} > n_k$ such that $|f_{n_{k+1}}(\alpha_j^{k+1}) - (-1)^j| < \delta$. Since $f_{n_{k+1}}$ is continuous, there are neighborhoods $U_{2i-1}^{k+1}, U_{2i}^{k+1} \subset U_i^k, i = 1, \ldots, 2^k$ such that for all $\alpha \in U_j^{k+1}$ we have $|f_{n_{k+1}}(\alpha) - (-1)^j| < \delta$.

In the m-th iteration this will give 2^m neighborhoods and m functions $f_{n_1}, \ldots, f_{n_m}$ so that for any $\epsilon_1, \ldots, \epsilon_m \in \{-1, +1\}$ there is an α contained in one of these neighborhoods such that $|f_k(\alpha) - \epsilon_k| < \delta$ for all $k = 1, \dots, m$. Hence

$$\|\sum_{k=1}^{m} \epsilon_k f_{n_k}\| \geq (1-\delta)m.$$

□

Theorem 1.3.9. *Let K be a compact metric space. Suppose there is an R-bounded commuting weakly compact approximating sequence in $C(K)$. Then $C(K)$ is isomorphic to c_0.*

Proof. By Corollary 1.3.6 we need only consider the case when K is countable. By Theorem 1.3.7 it suffices to consider the case when $K = [1, \alpha]$ where $\alpha \geq \omega^\omega$. Pick $f \in C(K)^{**}$ satisfying the hypotheses of Lemma 1.3.8.

Suppose (T_n) is an R-bounded weakly compact approximating sequence for $C(K)$. Then (T_n^*) is an approximating sequence for $C(K)^*$ by Theorem 1.3.5 and hence $T_n^{**}f$ converges to f weak*. It follows that for any m we can choose $n_1, \dots, n_m$ so that

$$\left\| \sum_{k=1}^{m} \epsilon_k T_{n_k}^{**} f \right\| \geq \tfrac{1}{2}m \qquad \epsilon_k = \pm 1.$$

Hence

$$\left(\mathbb{E} \left\| \sum_{k=1}^{m} \epsilon_k T_{n_k}^{**} f \right\|^2 \right)^{\frac{1}{2}} \geq \tfrac{1}{2}m.$$

This contradicts the fact that T_n is R-bounded (or even semi-R-bounded). □

Remark. We can replace the assumption of R-boundedness by the assumption that (T_n) and (T_n^*) are both semi-R-bounded. By Theorem 1.2.2 this hypothesis would imply that (T_n^*) is actually R-bounded and hence that (T_n) is WR-bounded. We only used the fact that (T_n) is both semi-R-bounded and WR-bounded.

Let us conclude by stating our main result with respect to semigroups. (Actually our results are somewhat stronger than stated below.)

Theorem 1.3.10. *Let X be a separable Banach space with an R-bounded strongly continuous semigroup $(T_t)_{t>0}$ consisting of weakly compact operators. Then if*

1. *$X = L_1(\mu)$ for some measure μ then X is isomorphic to ℓ_1 (i.e. μ is purely atomic).*
2. *$X = C(K)$ then X is isomorphic to c_0.*

Chapter 2

Real interpolation of domains of sectorial operators on L_p-spaces

2.1 Introduction

This chapter is based on joint work with L. Weis.

Let A be a sectorial operator on a non-atomic L_p-space, $1 \leq p < \infty$, whose resolvent consists of integral operators, or more generally, has a diffuse representation. Then the fractional domain spaces $\mathcal{D}(A^\alpha)$ for $\alpha \in (0,1)$ do not coincide with the real interpolation spaces of $(L_q, D(A))$. As a consequence, we obtain that no such operator A has a bounded H^∞-calculus if $p = 1$.

It is not uncommon that properties of the Laplace operator extend to a sectorial operator A which satisfies a pointwise kernel bound of the kind

$$|(\lambda + A)^{-1} f(u)| \leq \int\limits_\Omega k_\lambda(u,v)|f(v)|dv, \quad u \in \mathbb{R}^n \tag{2.1}$$

for $f \in L_q$ and λ in a sector about $\mathbb{R}_+$. Here, k_λ is the kernel of $(\lambda - \Delta)^{-1}$ or a more general Poisson bound. In the case of $1 < q < \infty$, (2.1) implies that $(-A)$ has maximal L_p-regularity for $1 < p < \infty$ (see e.g. [27], [35, section 5]), or that A has a H^∞-functional calculus on L_q if A has one on L_2 ([23], [35, section 5]). In this paper we exhibit two more examples of such phenomena.

It is well known that Laplace operator on $L_1(\mathbb{R}^n)$ does not have a bounded H^∞-calculus. In Corollary 2.3.3 we show that if q=1 then (2.1) implies that A does not have a bounded H^∞-functional calculus. This is still true if k_λ is the kernel of any positive integral operator on $L_1(\Omega)$ or if $(\lambda + A)^{-1}$ has a "diffuse representation" (see the definition below). If $(-A)$ generates a weakly compact semigroup this result is already contained in [28]. It seems remarkable that the very same estimate (2.1) that guarantees the boundedness of the H^∞-calculus in so many cases if $q \in (1, \infty)$, absolutely excludes it if $q = 1$.

It is also well known that for Δ on $L_q(\mathbb{R}^n), 1 < q < \infty, q \neq 2$ the fractional domains $D((1 - \Delta)^\alpha)$ are isomorphic to the Bessel potential spaces $W_q^{2\alpha}(\mathbb{R}^n)$. So they do not coincide with the real interpolation spaces $(L_q, \mathcal{D}(\Delta))_{\alpha,r}$ which are isomorphic to the Besov potential spaces $B_{q,r}^{2\alpha}(\mathbb{R}^n)$ (of course, they are the same for q=2). In Theorem 3.1 we will show that (2.1) implies such a result for any sectorial operator A on L_q with $0 \in \rho(A)$ and $1 < q < \infty, \quad q \neq 2$, i.e.

$$\mathcal{D}(A^\alpha) \neq (L_q, \mathcal{D}(A))_{\alpha,r}, \quad 0 < \alpha < 1$$

Again, it is enough that k_λ is the kernel of a positive integral operator on $L_q(\Omega)$, or that $(\lambda + A)^{-1}$ has a diffuse representation. If we assume in addition that A has bounded imaginary powers it follows that the complex and real interpolation methods yield different results for the interpolation pair $(L_q, \mathcal{D}(A))$ (see Corollary 2.3.2).

Let us recall now some definitions. A closed operator A with domain $\mathcal{D}(A)$ is called *sectorial of type* ω if the spectrum $\sigma(A)$ is contained in a sector $\{z \in \mathbb{C} : |arg(z)| < \omega\} \cup \{0\}$ and we have $\|\lambda R(\lambda, A)\| \leq C_\omega$ for $|arg(\lambda)| > \omega$. We will write $\rho(A) = \mathbb{C} \backslash \sigma(A)$ for the resolvent set of A and $R(\lambda, A)$ for the resolvent at $\lambda \in \rho(A)$. Suppose that A is a sectorial operator of type ω and f is a holomorphic function on Σ_σ where $\sigma > \omega$. Given that f satisfies the condition $\int_{\partial\Sigma_\delta} |f(\lambda)| \frac{1}{|\lambda|} |d\lambda| < \infty$, we can define

$$f(A) = \int\limits_{\partial\Sigma_\delta} f(\lambda) R(\lambda, A) d\lambda, \quad \omega < \delta < \sigma$$

We say that A has *bounded* $H^\infty(\Sigma_\sigma)$*-functional calculus* if the map $f \mapsto f(A)$ can be extended to a bounded map from the space $H^\infty(\Sigma_\sigma)$ of bounded holomorphic functions on Σ_σ to the space of bounded linear operators on X (see [32] for details).

For the definition of fractional powers in terms of the H^∞-calculus see e.g. [35] and if $0 \in \rho(A)$ see also [52]. A sectorial operator A has bounded imaginary powers if A^{-it} for $t \in \mathbb{R}$ define bounded operators on X. Clearly, a bounded H^∞-calculus implies bounded imaginary powers.

For the most part we consider L_q-spaces on σ-finite non-atomic measure spaces $(\mathfrak{K}, \mathfrak{B}, m)$ and (Ω, Σ, μ). We recall that a bounded operator T on L_q is *positive* if the image of every non-negative function is again a non-negative function. If an operator can be split into a difference of two positive operators then it is called *regular*. Regular operators between L_p spaces have a particularly useful representation (see [30, 54, 50]). Given a regular operator $T : L_p(K, m) \longrightarrow L_q(\Omega, \mu)$ there is a family of regular Borel measures $(\nu_y(x))_{y \in \Omega}$ on K such that for every $f \in L_p(K, m)$

we have

$$Tf(y) = \int_K f(x)\, d\nu_y(x) \qquad \mu - a.e.$$

Note that if all measures ν_y are absolutely continuous with respect to m then by the Radon-Nikodym theorem we obtain classical integral operators,

$$Tf(y) = \int_{\mathfrak{K}} f(x)k(y,x)\, dm(x), \qquad k(y,\cdot) = d\nu_y/dm$$

In case that all measures ν_y are non-atomic we say that the operator has a *diffuse representation.*

While resolvents of second order elliptic operators are typically classical integral operators, the resolvents of first order differential operators have usually a diffuse representation. As an example, consider the operator $A : D(A) \supset L_1(\mathbb{R}^2) \longrightarrow L_1(\mathbb{R}^2)$ given by

$$Af(x_1,x_2) = \frac{\partial}{\partial x_1} f(x_1,x_2)$$

Its resolvent

$$(R(A,\lambda)f)(x_1,x_2) = \int_0^\infty e^{-\lambda t} f(x_1+t,x_2)\, dt$$

has representing measure

$$\mu^\lambda_{(x_1,x_2)} = \eta^\lambda_{x_1} \otimes \delta_{x_2}$$

where δ_{x_2} is the Dirac measure and $d\eta^\lambda_{x_1} = \chi_{[x_1,\infty)}(t)e^{-\lambda(t-x_1)}dt$. Therefore, $R(A,\lambda)$ is not an integral operator but has a diffuse representation. However, given a diffuse operator T we can always pass to a sub-σ-algebra for which T is integral [55].

2.2 Preliminary results

The following lemma is a vector-valued version of a classical result about uniform integrability in L_1.

Lemma 2.2.1. *Let X be a Banach space and T be an isomorphic embedding from X into $L_p(X)$ $(1 \leq p < \infty)$. Assume that for some subspace $Y \subset X$ the set $\{\|Ty(t)\|_X^p : \quad y \in Y, \quad \|y\|_X = 1\}$ is not uniformly integrable as a subset of L_1. Then there exist a sequence (y_n) in Y isomorphic to a unit vector basis of l_p.*

Proof. Since $\{\|Ty(t)\|_X^p : \quad y \in Y, \quad \|y\|_X = 1\}$ is not uniformly integrable in L_1 we can find a sequence (y_n) in Y with $\|y_n\| \leq 1$ such that $\int \|Ty_n(t)\|_X^p dt = 1$ and $\|Ty_n(t)\|_X^p \longrightarrow 0\,(n \to \infty)$ almost everywhere. To see this, assume the contrary, i.e. every sequence from $T(Y)$ converging to zero almost everywhere is converging to zero in $L_p(X)$-norm. Then for all $0 < q < p$ there exists $C > 0$ such that $\int \|Ty(t)\|^p\, dt \leq C \int \|Ty(t)\|^q\, dt$ for all $y \in Y$. Hence, we have

$$\begin{aligned}
&\lim_{M\to\infty} \sup_{\|y\|=1} \Big(\int\limits_{\|Ty(t)\|>M} \|Ty(t)\|^p\, dt\Big)^{1/p} \\
&\leq C \lim_{M\to\infty} \sup_{\|y\|=1} \Big(\int\limits_{\|Ty(t)\|>M} \|Ty(t)\|^q\, dt\Big)^{1/q} \\
&\leq C \lim_{M\to\infty} \sup_{\|y\|=1} \Big(\int \|Ty(t)\|^p M^{q-p}\, dt\Big)^{1/p} = 0
\end{aligned}$$

This contradicts the fact that $\{\|Ty(t)\|_X^p : \quad y \in Y, \quad \|y\|_X = 1\}$ is not uniformly integrable in L_1.

For convenience define $f_n(t) = \|Ty_n(t)\|_X^p$. Then (f_n) are functions in L_1 of norm one. We will use a subsequence splitting lemma.

Lemma 2.2.2. *[29] If (f_n) is a sequence in the unit ball of L_1 then there exist a subsequence (f_{n_k}) and disjoint sets (A_k) with their complements B_k such that $f_{n_k}|_{B_k}$ are uniformly integrable.*

Since the sequence $(f_{n_k}|_{B_k})$ is uniformly integrable and still goes to zero almost everywhere when k is approaching infinity we get that $f_{n_k}|_{B_k}$ goes to zero in L_1-norm. So $f_{n_k}|_{A_k}$ is bounded in norm from below. Now $Ty_{n_k} = Ty_{n_k}|_{B_k} + Ty_{n_k}|_{A_k}$ where $\|Ty_{n_k}|_{A_k}\|_{L_p(X)} = \|f_{n_k}|_{A_k}\|_{L_1}$ is bounded from below. Thus the sequence $(Ty_{n_k}|_{A_k})$ is isomorphic to the unit vector basis of l_p since it has disjoint support and bounded from below in $L_p(X)$. On the other hand

$$\|Ty_{n_k} - Ty_{n_k}|_{A_k}\|_{L_p(X)} = \|Ty_{n_k}|_{B_k}\|_{L_p(X)} = \|f_{n_k}|_{B_k}\|_{L_1} \longrightarrow 0 \quad (k \to \infty)$$

It follows by perturbation of basis that some subsequence of (Ty_{n_k}) is equivalent to the unit vector basis of l_p. Denote this subsequence again by (Ty_{n_k}). Then (y_{n_k}) is also equivalent to the unit vector basis of l_p since T is an isomorphism. $\square$

The next proposition is related to a result in [32]. The expression appearing in the statement will be applied to the setting of interpolation spaces between X and $D(A)$.

Proposition 2.2.3. *Suppose X is a Banach space and A is a sectorial operator on X. Assume there is a constant $C > 0$, $1 \le p < \infty$ and $\alpha \in (0,1)$ such that for every $x \in X$*

$$C^{-1}\|x\| \le \Big(\int_{-\infty}^{0} \||t|^{\alpha-1/p} A^\alpha R(t,A)x\|^p dt\Big)^{1/p} \le C\|x\| \tag{2.2}$$

Then if Y is an infinite-dimensional closed subspace of $D(A)$ (with a graph norm) and Y does not contain a copy of l_p then A is bounded on Y.

Proof. We will consider an operator $T : X \mapsto L_p(\mathbb{R}_-, dt, X)$ given by

$$Tx(t) = |t|^{\alpha - 1/p} A^\alpha R(t, A)x$$

It follows from (2.2) that T is an isomorphic embedding. Since $\alpha < 1$ we can find a natural number m such that $\alpha \leq (m-1)/m$. Fix $s < 0$. Then $R(s, A)$ maps X isomorphically onto $D(A)$ (with a graph norm). Let $Y_0 = R(s, A)^{-1}Y$. Then Y_0 is an infinite-dimensional subspace of X that does not contain a copy of l_p. By lemma 2.2.1 the set $\{\|Ty(t)\|_X^p : \quad y \in Y_0, \quad \|y\|_X = 1\}$ is uniformly integrable. The operator $A^\alpha R(s, A)$ has a lower bound on Y_0 since otherwise, there would exist a sequence y_n in Y_0 of elements of norm one such that $\|A^\alpha R(s, A)y_n\| \to 0$. However, the resolvent equation yields for any $t < 0$

$$A^\alpha R(t, A)y_n = A^\alpha R(s, A)y_n + (s - t)R(t, A)(A^\alpha R(s, A)y_n)$$

Therefore $\|A^\alpha R(t, A)y_n\| \to 0$ pointwise. Now by uniform integrability and 2.2, we have $\|y_n\| \to 0$ which gives a contradiction.

The operator $A^\alpha R(s, A)$ is an isomorphism on Y_0. Thus the subspace $Y_1 = A^\alpha R(s, A)(Y_0)$ does not contain a copy of l_p and by the same argument we get that $A^\alpha R(s, A)$ is bounded from below on Y_1. This gives us a lower bound for the operator $A^{2\alpha} R(s, A)^2$ on Y_0. Repeating the same procedure m times we get that the operator $A^{m\alpha} R(s, A)^m$ is bounded from below on Y_0 by some constant $C > 0$. It follows from the boundedness of the operator $A^{m\alpha} R(s, A)^{m-1} \quad (\alpha \leq (m-1)/m)$

and the simple computation

$$C\|y_0\| \leq \|A^{m\alpha}R(s,A)^m y_0\| \leq \|A^{m\alpha}R(s,A)^{m-1}\| \; \|R(s,A)y_0\| \quad y_0 \in Y_0$$

that the resolvent $R(s,A)$ is bounded from below on Y_0.

Now we see that A is bounded on $Y = R(s,A)Y_0$. Take any y in Y and find y_0 in Y_0 such that $y = R(s,A)y_0$. Then

$$\begin{aligned}\|Ay\| &\leq \|AR(s,A)\| \; \|y_0\| \\ &\leq (1/C)\|AR(s,A)\| \; \|A^{m\alpha}R(s,A)^{m-1}\| \; \|R(s,A)y_0\| \\ &= C_1\|y\|\end{aligned}$$

□

Remark 2.2.4. The proposition cannot be applied for $p = 2$. In this case X is isomorphic to $L_2(\mathbb{R}_-, dt, X)$. Thus there is no subspace in X and hence in $\mathcal{D}(A)$ which does not contain a copy of l_2.

We assume that zero is contained in the resolvent set, then $(-\infty, 0] \subset \rho(A)$ and we have an estimate $\|R(t,A)\| \leq \frac{C}{1+|t|}$ for all $t \in (-\infty, 0]$. This allows us to apply a theorem from [52] which yields that an equivalent norm on the real interpolation space $(X, \mathcal{D}(A))_{\alpha,p}$ for $0 < \alpha < 1$ and $1 \leq p \leq \infty$ is given by

$$\|x\|_{(L_q,D(A))_{\alpha,p}} \approx (\int_0^\infty \|t^\alpha A(A+t)^{-1}x\|^p \frac{dt}{t})^{1/p} \tag{2.3}$$

for $x \in (X, \mathcal{D}(A))_{\alpha,p}$.

In [32] it was shown that if A has an H^∞-calculus on L_1 then

$$\|x\|_{L_1} \approx \int_{-\infty}^{\infty} \|A^s R(t,A)x\| \, \frac{dt}{t}.$$

Formula 2.3 allows us to reformulate this statement as follows.

Proposition 2.2.5. *If A has a bounded H^∞-calculus on $L_1(\Omega, \mu)$ then $(L_1, \mathcal{D}(A))_{\alpha,1} = \mathcal{D}(A)^\alpha$ with equivalence of norms for $0 < \alpha < 1$.*

2.3 Main results

In general we have the following inclusions between the domain $\mathcal{D}(A^\alpha)$ of a fractional power of A and real interpolation spaces $(X, \mathcal{D}(A))_{\alpha,1}$ and $(X, \mathcal{D}(A))_{\alpha,\infty}$

$$(X, \mathcal{D}(A))_{\alpha,1} \subset \mathcal{D}(A^\alpha) \subset (X, \mathcal{D}(A))_{\alpha,\infty}.$$

If a sectorial operator A has a bounded H^∞-calculus on $X = L_2(\mathfrak{K}, \mathfrak{B}, m)$ then we have $\mathcal{D}(A^\alpha) = (X, \mathcal{D}(A))_{\alpha,2}$. This result can be found in [**?**]. As we will see now this statement is wrong for L_q with $q \neq 2$.

Theorem 2.3.1. *Let A be a sectorial operator on $L_q(\mathfrak{K}, \mathfrak{B}, m)$ for a non-atomic measure space $(\mathfrak{K}, \mathfrak{B}, m)$ and $1 \leq q < \infty, q \neq 2$. Assume that $0 \in \rho(A)$ and there exists $s < 0$ such that $R(s, A)$ is a regular operator with a diffuse representation. Then for any $\alpha \in (0, 1)$ and $1 \leq p \leq \infty$*

$$D(A^\alpha) \neq (L_q, D(A))_{\alpha,p}$$

Proof. We will assume that $\mathcal{D}(A^\alpha) = (L_q, \mathcal{D}(A^\alpha))_{\alpha,p}$ and derive a contradiction.

It follows from [52] that there exists a constant $C > 0$ such that for any $y \in$

$(L_q, D(A))_{\alpha,p}$ we have

$$C^{-1}\Big(\int_0^\infty \|t^\alpha A(A+t)^{-1}y\|^p \frac{dt}{t}\Big)^{1/p} \leq \|y\|_{(L_q,D(A))_{\alpha,p}}$$

$$\leq C\Big(\int_0^\infty \|t^\alpha A(A+t)^{-1}y\|^p \frac{dt}{t}\Big)^{1/p}$$

Since $D(A^\alpha) = (L_q, D(A))_{\alpha,p}$, we obtain for any $y \in D(A^\alpha)$ that the quantities $\|A^\alpha y\|, \quad \|y\|_{D(A^\alpha)}$, and $\|y\|_{(L_q,D(A))_{\alpha,p}}$ are equivalent.

Pick x from the range of A^α and take $y \in \mathcal{D}(A^\alpha)$ such that $x = A^\alpha y$. Then using $\int_0^\infty \|t^\alpha A(A+t)^{-1}A^{-\alpha}x\|^p \frac{dt}{t} = \int_{-\infty}^0 \||t|^{\alpha-1/p}A^{1-\alpha}R(t,A)x\|^p dt$ we obtain for some suitable constant $C > 0$ that

$$C^{-1}\Big(\int_{-\infty}^0 \||t|^{\alpha-1/p}A^{1-\alpha}R(t,A)x\|^p dt\Big)^{1/p} \leq \|x\|_{L_q(\mathfrak{K},\mathfrak{B},m)}$$

$$\leq C\Big(\int_{-\infty}^0 \||t|^{\alpha-1/p}A^{1-\alpha}R(t,A)x\|^p dt\Big)^{1/p}$$

The range of A^α is dense in $L_q(\mathfrak{K},\mathfrak{B},m)$ and therefore condition (2.2) is fulfilled. We will use Proposition 2.2.3. Since the resolvent $R(s,A)$ is a regular operator with a diffuse representation there is a non-atomic sub σ-algebra $\mathfrak{B}_1$ of $\mathfrak{B}$ such that $R(s,A)|_{L_q(K,\mathfrak{B}_1,m)}$ is a compact operator ([54]). Let Y_1 be a closed infinite-dimensional subspace of $L_q(\mathfrak{K},\mathfrak{B}_1,m)$ which does not contain a copy of l_p, for instance, take the span of a sequence equivalent to the Rademacher functions. Consider $Y = R(s,A)Y_1$. Since $R(s,A)$ is an isomorphism from $L_q(\mathfrak{K},\mathfrak{B},m)$ onto $D(A)$ (with the graph norm), Y is a closed infinite-dimensional subspace of $D(A)$ and does not contain l_p. By Proposition 2.2.3 A is bounded on Y and therefore

$sI - A$ is also bounded on Y. We consider the bounded operator

$$J : (D(A), \|.\|_A) \longrightarrow L_q(\mathfrak{K}, \mathfrak{B}, m), \quad J = R(s, A)(sI - A)$$

Then $J(Y) = Y$. On the other hand, $J|_Y = R(s, A)(sI - A)|_Y = R(s, A)|_{Y_1}$ is a compact operator since $Y_1 \subset L_q(\mathfrak{K}, \mathfrak{B}_1, m)$. This is impossible since J is onto Y and Y is infinite-dimensional. We hence obtain a contradiction. □

It is well known that if A has bounded imaginary powers on X then $\mathcal{D}(A^\alpha)$ coincides with the complex interpolation spaces $[X, \mathcal{D}(A)]^\alpha = \mathcal{D}(A)^\alpha$ (see e.g. [35] [52]). Hence our theorem implies

Corollary 2.3.2. *Assume in addition to the assumption of Theorem 2.3.1 that A has bounded imaginary powers. Then*

$$(L_p, \mathcal{D}(A))_{\alpha,p} \neq [L_p, \mathcal{D}(A)]_\alpha$$

for all $1 \leq p \leq \infty$ *and* $\alpha \in (0, 1)$.

Our next results will show that no reasonable differential operator on $L_1(\Omega, \mu)$ can have a bounded H^∞-calculus.

Corollary 2.3.3. *Let* A *be a sectorial operator on* $L_1(\Omega, \Sigma, \mu)$. *Assume there is a point* $\lambda \in \rho(A)$ *such that the resolvent* $R(\lambda, A)$ *has a diffuse representation. Then* A *does not have a bounded* H^∞*-calculus.*

Proof. Combine Proposition 2.2.5 and Theorem 2.3.1 noticing that all operators on L_1 are regular. □

For a variant of our assumption recall the Sobolev spaces defined for $s \in \mathbb{R}$ and $1 \leq p \leq \infty$ as

$$H_p^s = \{u \in \mathcal{S}' : \|\mathcal{F}^{-1}\{(1 + |\xi|^2)^{s/2}\mathcal{F}u(\xi)\}\|_{L_p} < \infty.$$

where $\mathcal{F} : \mathcal{S}' \longrightarrow \mathcal{S}'$ denotes the Fourier transform for tempered distributions (see [5], [52]).

Corollary 2.3.4. *Let $\Omega \subset \mathbb{R}^n$ with piecewise smooth boundary. Suppose that $A : L_1(\Omega) \supset \mathcal{D}(A) \longrightarrow L_1(\Omega)$ is a sectorial operator such that $\mathcal{D}(A) \subset H_1^s(\Omega)$ for some $s > 0$. Then A does not have an H^∞-calculus.*

Proof. To apply Theorem 2.3.3 we need to show that $R(\lambda, A)$ has a diffuse representation for some $\lambda \in \rho(A)$. Pick any $\lambda \in \rho(A)$. Then by Sobolev's theorem we have a continuous inclusion $H_1^s(\Omega) \hookrightarrow L_p(\Omega)$ for some $p > 1$. Hence, for any bounded set $U \subset \Omega$ with piecewise smooth boundary we obtain that $\chi_U R(\lambda, A)$ factors through $L_p(U)$,

$$L_1(\Omega) \overset{\chi_U R(\lambda,A)}{\longrightarrow} \mathcal{D}(A) \cap L_1(U) \hookrightarrow H_1^s(U) \hookrightarrow L_p(U) \hookrightarrow L_1(U).$$

Consequently, $\chi_U R(\lambda, A)$ is a weakly compact operator. Notice that $\mu(U)$ is finite. Therefore, $\chi_U R(\lambda, A)$ is an integral operator [20]. This argument works for all bounded $U \subset \Omega$ with piecewise smooth boundary and thus $R(\lambda, A)$ has a diffuse representation. According to Corollary 2.3.3, A does not have an H^∞-calculus. $\square$

Chapter 3

Rademacher bounded families of operators on L_1

3.1 Introduction

Recall that a closed operator A on a Banach space X is called sectorial if

- The domain $\mathcal{D}(A)$ and range $\mathbb{R}(A)$ are dense,
- A is one-to-one
- The spectrum $\sigma(A)$ is contained in a sector $\Sigma_\omega = \{\zeta : |\arg \zeta| \leq \omega\}$.
- The resolvent $R(\zeta, A)$ satisfies the estimate

  $$\|\zeta R(\zeta, A)\| \leq C_\phi,\ |arg(\zeta)| \geq \phi$$

  whenever $\omega < \phi < \pi$.

If $\omega < \frac{\pi}{2}$ than the operator $-A$ generates a bounded analytic semigroup, $T_u = e^{-uA}, u \in [0, 1]$. Conversely if $-A$ is the generator of a bounded analytic semigroup then A is sectorial, provided it is one-one with $\omega < \pi/2$.

A is called R-sectorial with angle of R-sectoriality $\omega_R = \omega_R(A)$ if for every $\phi > \omega_R$ the collection of operators $\{R(\zeta, A) : \ |\arg \zeta| \geq \phi\}$ is R-bounded. We recall here that a collection of operators $\mathcal{T}$ on a Banach space X is called R-bounded if there is a constant C so that

$$(\mathbb{E}\|\sum_{j=1}^{n}\varepsilon_j T_j x_j\|^2)^{1/2} \leq C(\mathbb{E}\|\sum_{j=1}^{n}\varepsilon_j x_j\|^2)^{1/2} \qquad x_1,\ldots,x_n \in X,\ T_1,\ldots,T_n \in \mathcal{T}.$$

Here $(\varepsilon_j)_{j=1}^{\infty}$ is a sequence of independent Rademacher functions. R-sectoriality has become a very important property for sectorial operators because of the theorem of Weis [53] which shows that in a Banach space with (UMD) this property is closely related to maximal regularity.

This note is concerned with the structure of R-sectorial operators on the Banach space $L_1 = L_1(K,\lambda)$ where K is a Polish space and λ is a nonatomic $\sigma-$finite Borel measure. All such spaces are isometric to $L_1 = L_1[0,1]$, and so we will assume that K is a compact metric space and λ is a probability measure.

If A is a sectorial operator on L_1 which has H^∞-calculus (for some angle) then A is R-sectorial (for some angle). We refer to [32] for the definition and discussion of the $H^\infty-$calculus. In [32] it was shown that if A has an $H^\infty-$calculus then A is bounded on any reflexive subspace of $\mathcal{D}(A)$ (with the graph norm); this had the implication there are very few examples of sectorial operators with an $H^\infty-$calculus on L_1 and in particular essentially no reasonable differential operator can have this property. In [28] it was shown that there are no R-bounded strongly continuous semigroups on L_1 consisting of weakly compact operators; it also follows from the results of [28] that if A is an R-sectorial operator on L_1 then the resolvent $R(\zeta, A)$ can never be a weakly compact operator.

The simplest example of a sectorial operator on $L_1(K,\lambda)$ which has an $H^\infty-$calculus and hence is R-sectorial is the operator

$$Af(s) = b(s)f(s)$$

where $b > 0$ a.e. and with domain

$$\mathcal{D}(A) = \left\{ f : \int |f(s)| b(s)^{-1} d\lambda(s) < \infty \right\}.$$

Note here that the domain is very large indeed; in fact for any $\epsilon > 0$ we can find a Borel set B with $\lambda(B) > 1-\epsilon$ and such that $L_1(B) \subset \mathcal{D}(A)$. Of course one can get further examples by considering $A' = UAU^{-1}$ for U any invertible operator with $\mathcal{D}(A') = U(\mathcal{D}(A))$.

In this note, we show that this example is typical. Precisely we show that if A is R-sectorial and $\epsilon > 0$ then there is an invertible operator $U : L_1 \to L_1$ with $\|U-I\| < \epsilon$ such that for some positive Borel function w we have $U(\mathcal{D}(A)) \supset L_1(w)$. This improves both the results of [28] and [32].

3.2 Operators on L_1

Let K be a compact metric space and suppose λ is a probability measure on K. We will utilize the so-called random measure representation of operators on L_1, developed in [30], [24] and [54].

A random measure on K is a map $s \to \mu_s$ from K into $\mathcal{M}(K)$ which is Borel for the weak*-topology on $\mathcal{M}(K)$. If the random measure satisfies the condition

$$\int_K |\mu_s|(B) d\lambda(s) \leq C\lambda(B) \qquad B \in \mathcal{B} \tag{3.1}$$

then it induces a bounded operator $T : L_1(\lambda) \to L_1(\lambda)$ given by the formula

$$Tf(s) = \int_K f(t) d\mu_s(t) \qquad \lambda - \text{a.e.} \tag{3.2}$$

and then $\|T\| \leq C$.

Conversely every bounded linear operator $T : L_1(\lambda) \to L_1(\lambda)$ has an essentially unique random measure representation and $\|T\|$ is the least constant C so that (3.1) holds.

We may also associate to T a unique measure ρ_T on $K \times K$ given by

$$\rho_T(A) = \int_K \left(\int_K \chi_A(s,t) d\mu_s^T(t) \right) d\lambda(s) \qquad A \in \mathcal{B}(K \times K).$$

Thus

$$\rho_T(A \times B) = \int_A T\chi_B d\lambda.$$

The map $T \to \rho_T$ maps $\mathcal{L}(L_1)$ onto an order-ideal in $\mathcal{M}(K \times K)$ consisting of all measures ρ such that

$$|\rho|(A \times B) \leq C\lambda(B), \qquad A, B \in \mathcal{B}(K \times K).$$

The space $\mathcal{L}(L_1(K, \lambda))$ is a complex Banach lattice and it is easily checked that if $T \in \mathcal{L}(L_1)$ then $\mu_s^{|T|} = |\mu_s^T|$ (λ-a.e.) and that $\rho_{|T|} = |\rho_T|$. Since it is a Banach lattice we can define as usual an operator $(\sum_{j=1}^n |T_j|^2)^{\frac{1}{2}}$ for any $T_1, \ldots, T_n \in X$.

The following result is implicitly contained in ideas of [30], and more explicitly in [31] :

Proposition 3.2.1. *Let* $T_n : L_1 \to L_1$ *be a uniformly bounded sequence of operators such that* $\lim_{n\to\infty} \|\rho_{T_n}\| = 0$. *Then given any* $\epsilon > 0$ *there is a Borel subset* B *of* K

with $\lambda(B) > 1 - \epsilon$ *and* $n \in \mathbb{N}$ *so that we have*

$$\|T_n f\| \le \epsilon \|f\| \qquad f \in L_1(B).$$

Proof. Consider the measures on K given by

$$\nu_n(A) = |\rho_{T_n}|(A \times K).$$

Then ν_n is absolutely continuous with respect to λ. Let w_n be the Radon-Nikodym derivatives. Then

$$\int w_n d\lambda = |\rho_{T_n}|(K \times K) \to 0.$$

Therefore, $w_n \longrightarrow 0$ in measure. Hence there exists $n \in \mathbb{N}$ and B with $\lambda(B) > 1-\epsilon$ so that $|w_n| < \epsilon$ on B.

However

$$\|T_n\| = \|w_n\|_\infty.$$

This follows from $\|T_n\| = \sup_{\lambda(B)>0} \frac{v_n(B)}{\lambda(B)}$, $v_n(B) = \int_B w_n \, d\lambda$, and $v_n(B) = |\rho_{T_n}|(B \times K) = \iint \chi_{B\times K} \, d|\mu_s^{T_n}|(t) = \int_K |\mu_s^{T_n}|(B) \, d\lambda$.

If $f \in L_1(B)$ we have

$$\|T_n f\| \le \int_{K\times K} |f(s)| d|\rho_T|(s,t) \le \int_B |f(s)| w_n(s) d\lambda(s) \le \epsilon \|f\|.$$

□

If $T \in \mathcal{L}(L_1)$ then we can write μ_s as given in (3.2) in the form

$$\mu_s = a(s)\delta_s + \mu_s' \qquad \lambda - \text{a.e.}$$

where $\mu_s'\{s\} = 0$ λ–a.e. and a is a bounded Borel function. (See for example [30]). Thus

$$Tf(s) = a(s)f(s) + \int_K f(t) d\mu_s'(t) \qquad \lambda - \text{a.e.}$$

If we define the diagonal part of T by

$$\Pi(T)f = a(s)f(s)$$

then $\rho_{\Pi(T)}$ is the restriction of ρ_T to the diagonal subset $\Delta = \{(s,s): \ s \in K\}$. Thus

$$\rho_{\Pi(T)}(B) = \rho_T(B \cap \Delta).$$

Theorem 3.2.2. *Let $\mathcal{T}$ be a family of operators in $\mathcal{L}(L_1)$. Then the following are equivalent:*

(i) $\mathcal{T}$ is $R-$bounded.

(ii) $\{(\sum_{j=1}^n a_j^2 |T_j|^2)^{1/2} : \sum_{j=1}^n |a_j|^2 \leq 1, T_k \in \mathcal{T}, n \in \mathbb{N}\}$ is uniformly bounded.

Proof. Assume $\mathcal{T}$ is R-bounded so that

$$\mathbb{E}\|\sum_{j=1}^n \epsilon_j T_j x_j\| \leq M\mathbb{E}\|\sum_{j=1}^m \epsilon_j x_j\|$$

for any $T_1, \ldots, T_n \in \mathcal{T}$ and $x_1, \ldots, x_n \in X$. Suppose $T_1, \ldots T_n \in \mathcal{T}$ and $a_1, \ldots, a_n \in \mathbb{C}$ are such that $\sum_{j=1}^n |a_j|^2 \leq 1$. Then, by Khintchine's inequality for lattices,

$$\left\|(\sum_{j=1}^n |a_j|^2 |T_j|^2)^{\frac{1}{2}}\right\| \leq C\mathbb{E}\left\||\sum_{j=1}^n \epsilon_j a_j T_j|\right\|$$

where C is an absolute constant.

Now choose any sequence of partitions $\mathcal{A}_m = (A_{mj})_{j=1}^{N_m}$ so that each $\mathcal{A}_{m+1}$ refines $\mathcal{A}_m$ and

$$\lim_{m\to\infty} \sup_{1\leq j\leq N_m} \operatorname{diam} A_{mj} = 0.$$

Then for any positive function $f \in L_1(K,\lambda)$ and any $T \in \mathcal{L}(L_1(K,\lambda))$ we have

$$|T|f = \lim_{m\to\infty} \sum_{j=1}^{N_m} |T(f\chi_{A_{mj}})| \qquad \lambda - \text{a.e.}$$

Thus

$$|\sum_{k=1}^{n} \epsilon_k a_k T_k| f = \lim_{m\to\infty} \sum_{j=1}^{N_m} |\sum_{k=1}^{n} \epsilon_k a_k T_k(f\chi_{A_{mj}})| \qquad \lambda - \text{a.e.}$$

Now by R-boundedness

$$\begin{aligned}\mathbb{E}\int_K \sum_{j=1}^{N_m} |\sum_{k=1}^{n} \epsilon_k a_k T_k(f\chi_{A_{mj}})| d\lambda &= \sum_{j=1}^{N_m} \mathbb{E}\| \sum_{k=1}^{n} \epsilon_k a_k T_k(f\chi_{A_{mj}})\| \\ &\leq M \sum_{j=1}^{N_m} \mathbb{E}\| \sum_{k=1}^{n} \epsilon_k a_k f\chi_{A_{mj}}\| \\ &\leq M(\sum_{k=1}^{n} |a_k|^2)^{\frac{1}{2}} \sum_{j=1}^{N_m} \|f\chi_{A_{mj}}\| \\ &\leq M\|f\|.\end{aligned}$$

It follows from Fatou's Lemma that

$$\mathbb{E}\| \sum_{k=1}^{n} \epsilon_k a_k T_k\| \leq M$$

and hence

$$\|(\sum_{k=1}^{n} |a_k|^2 |T_k|^2)^{\frac{1}{2}}\| \leq CM.$$

Now let us prove that $(ii) \implies (i)$. First suppose $f \in L_1(K,\lambda)$ is positive and $T_1, \dots, T_n \in \mathcal{L}(L_1(K,\lambda))$. Then if $a_1, \dots, a_n \geq 0$ and $a_1^2 + \cdots + a_n^2 = 1$ we have

$$\sum_{j=1}^{n} a_j |T_j| f \leq (\sum_{j=1}^{n} |T_j|^2)^{\frac{1}{2}} f.$$

Now the least upper bound of the left hand-side over all choices of $a_1, \dots, a_n$ is $(\sum_{j=1}^{n} (|T_j|f)^2)^{\frac{1}{2}}$ and so

$$(\sum_{j=1}^{n} (|T_j|f)^2)^{\frac{1}{2}} \leq (\sum_{j=1}^{n} |T_j|^2)^{\frac{1}{2}} f.$$

Let from our assumption (ii) there is a constant M so that

$$\|(\sum_{j=1}^{n}|a_j|^2|T_j|^2)^{\frac{1}{2}}\| \le M \qquad T_1,\dots,T_n \in \mathcal{T},\ |a_1|^2+\cdots+|a_n|^2=1.$$

Suppose $f \in L_1$ and $T_1,\dots,T_n \in \mathcal{T}$.

Then

$$\begin{aligned}\mathbb{E}\|\sum_{j=1}^{n}\epsilon_j a_j T_j f\| &\le \|(\sum_{j=1}^{n}|a_j|^2|T_jf|^2)^{\frac{1}{2}}\| \\ &\le \|(\sum_{j=1}^{n}|a_j|^2(|T_j||f|)^2)^{\frac{1}{2}}\| \\ &\le \|(\sum_{j=1}^{n}|a_j|^2|T_j|^2)^{\frac{1}{2}}|f|\| \\ &\le M\|f\|.\end{aligned}$$

Now by Theorem 2.2 of [28] this means $\mathcal{T}$ is R-bounded. □

Proposition 3.2.3. *Suppose $\mathcal{T}$ is an R-bounded family of operators on $L_1(K,\lambda)$. Then the family of measures $\{\rho_T :\ T \in \mathcal{T}\}$ is relatively weakly compact in $\mathcal{M}(K\times K)$.*

Proof. Note that if $T_1,\dots,T_n \in \mathcal{T}$ then

$$\|\max_{1\le k\le n}|T_k|\| \le \|(\sum_{k=1}^{n}|T_k|^2)^{\frac{1}{2}}\| \le Mn^{1/2}$$

where

$$M=\sup\{\|\sum_{j=1}^{n}|a_j|^2|T_j|^2)^{\frac{1}{2}} :\ T_1,\dots,T_n\in\mathcal{T},\ \sum_{j=1}^{n}|a_j|^2\le 1\}$$

which is finite by Theorem 3.2.2. The maximum here is computed in the lattice $\mathcal{L}(L_1)$.

Hence

$$\| \max_{1\leq k\leq n} |\rho_{T_k}| \, \|_{\mathcal{M}(K\times K)} \leq Mn^{\frac{1}{2}}.$$

Assume the set $\{\rho_T : T \in \mathcal{T}\}$ is not relatively weakly compact. Then by Dieudonne-Grothendieck characterization [18] there is a $\delta > 0$, a sequence $(T_k)_{k=1}^n$ and a sequence of disjoint open sets U_k in $K \times K$ so that $\rho_{T_k}(U_k) \geq \delta$ for all k. Then

$$\| \max_{1\leq k\leq n} |\rho_{T_k}| \, \|_{\mathcal{M}(K\times K)} \geq \sum_{k=1}^{n} \rho_{T_k}(U_k) \geq \delta n \qquad n = 1, 2, \ldots$$

which gives a contradiction. □

3.3 Applications to sectorial operators

Now we will give some applications of this theorem to sectorial operators.

Proposition 3.3.1. *If A is R-sectorial and $\omega_R(A) < \pi/2$ then $\{e^{-tA} : 0 < t < \infty\}$ is an R-bounded semigroup. Conversely if A is sectorial and $-A$ generates an R-bounded semigroup then A is R-sectorial with $\omega_R(A) \leq \pi/2$.*

If $-A$ is a sectorial operator which generates a semigroup $\{e^{-tA} : 0 < t < \infty\}$ with the property that $\{e^{-tA} : 0 < t \leq 1\}$ is R-bounded then for any $\phi > \pi/2$ there exists M so that the set $\{\zeta R(\zeta, A) : \, |\arg(\zeta + M)| \geq \phi\}$ is R-bounded.

Proof. These are simple deductions from the formulas:

$$\zeta R(\zeta, A) = \int_0^\infty \zeta e^{\zeta t} e^{-tA} dt$$

and

$$e^{-tA} - (1 + tA)^{-1} = -\frac{1}{2\pi i} \int_{\Gamma_\nu} (e^{-t\zeta} - (1 + t\zeta)^{-1}) R(\zeta, A) d\zeta$$

where Γ_ν is a contour of the form $\{|s|e^{i(\text{sgn } s)\nu} : -\infty < s < \infty\}$ for any ν with $\nu > \omega(A)$. Indeed, assuming that $\{e^{-At} : 0 < t < \infty\}$ is R-bounded we fix some angle $\frac{\pi}{2} < \varphi < \pi$. Then for any choice of numbers $\zeta_j = r_j e^{i\varphi_j}$ with $\varphi_j \geq \varphi$, $j = 1, ..., n$ we obtain

$$\begin{aligned}(\mathbb{E}\|\sum_{j=1}^n \epsilon_j \zeta_j R(\zeta_j, A)x_j\|^2)^{\frac{1}{2}} &= (\mathbb{E}\|\sum_{j=1}^n \int_0^\infty \epsilon_j r_j e^{i\varphi_j} e^{tr_j e^{i\varphi_j}} e^{-tA} x_j dt\|^2)^{\frac{1}{2}} \\ &\leq (\int_0^\infty \mathbb{E}\|\sum_{j=1}^n \epsilon_j e^{i\varphi_j} e^{se^{i\varphi_j}} e^{-s/r_j A} x_j\|^2 ds)^{\frac{1}{2}} \\ &\leq C(\int_0^\infty \max_j |e^{se^{i\varphi_j}}|^2 ds)^{\frac{1}{2}} (\mathbb{E}\|\sum_{j=1}^n \epsilon_j x_j\|^2)^{\frac{1}{2}} \\ &\leq \frac{C}{|\cos\varphi|} (\mathbb{E}\|\sum_{j=1}^n \epsilon_j x_j\|^2)^{\frac{1}{2}}.\end{aligned}$$

Therefore A is R-sectorial with sectoriality angle $\omega_R(A) \leq \pi/2$. Similarly, it follows that if A is R-sectorial and $\omega_R(A) < \pi/2$ then $\{e^{-tA} : 0 < t < \infty\}$ is R-bounded.

For the last statement suppose that C is a constant such that

$$(\mathbb{E}\|\sum_{j=1}^n \varepsilon_j e^{-t_j A} x_j\|^2)^{\frac{1}{2}} \leq C(\mathbb{E}\|\sum_{j=1}^n \varepsilon_j x_j\|^2)^{\frac{1}{2}}$$

whenever $x_1, \ldots, x_n \in X,\ 0 \leq t_1, \ldots, t_n \leq 1$.

Then if $m \in \mathbb{N}$,

$$(\mathbb{E}\|\sum_{j=1}^n \varepsilon_j e^{-(m+t_j)A} x_j\|^2)^{\frac{1}{2}} \leq CK^m(\mathbb{E}\|\sum_{j=1}^n \varepsilon_j x_j\|^2)^{\frac{1}{2}}$$

where $K = \|e^{-A}\|$. Now we show that the set $\{e^{-ut}e^{-tA} : 0 < t < \infty\}$ is R-bounded as long as $e^u > K$. For $x_1, ..., x_n \in X$ and $0 < t_1, ...t_n < \infty$ we obtain

$$\begin{aligned}(\mathbb{E}\|\sum_{j=1}^n \varepsilon_j e^{-t_j u} e^{-t_j A} x_j\|^2)^{\frac{1}{2}} &= (\mathbb{E}\|\sum_{m=0}^\infty \sum_{m \leq t_j < m+1} \varepsilon_j e^{-t_j u} e^{-t_j A} x_j\|^2)^{\frac{1}{2}} \\ &\leq C\sum_{m=0}^\infty K^m e^{-um} (\mathbb{E}\|\sum_{m \leq t_j < m+1} \varepsilon_j e^{-u\tilde{t_j}} x_j\|^2)^{\frac{1}{2}}\end{aligned}$$

where $0 \leq \tilde{t_j} \leq 1$. By the contraction principle

$$(\mathbb{E}\|\sum_{m\leq t_j<m+1} \varepsilon_j e^{-\tilde{t_j}} x_j\|^2)^{\frac{1}{2}} \leq \max_{1\leq j\leq n} |e^{-u\tilde{t_j}}|(\mathbb{E}\|\sum_{j=1}^{n} \varepsilon_j x_j\|^2)^{\frac{1}{2}} \leq (\mathbb{E}\|\sum_{j=1}^{n} \varepsilon_j x_j\|^2)^{\frac{1}{2}}$$

Since $\sum_{m=0}^{\infty} K^m e^{-um}$ is finite for $u > \ln K$ we obtain the claim. Consequently, the set $\{\xi R(\xi, u + A) : |\arg \xi| > \phi\}$ is R-bounded.

Now for $\zeta \in \mathbb{C}$ with $|\arg(\zeta + M)| > \phi$, $M > u$ we can rewrite using $\xi - u = \zeta$,

$$\zeta R(\zeta, A) = (\xi - u)R(\xi - u) = (\xi - u)R(\xi, A + u) = \frac{\xi - u}{\xi}\xi R(\xi, A + u)$$

Since $|\frac{\xi-u}{\xi}| \leq \frac{M}{M-u}$ the result follows quickly. $\square$

It was shown in [32] that an operator with H^∞-calculus on L_1 is already R-sectorial. Moreover, there are no R-bounded strongly continuous semigroups on L_1 consisting of weakly compact operators [28], and if $-A$ generates a weakly compact semigroup (e^{-uA}) then the resolvents $R(\xi, A)$ are necessarily weakly compact [54]. Now, in the case where A is a sectorial operator which generates an R-bounded semigroup on L_1, we will go even further and show that the resolvents have a representation quite unlike a weakly compact operator. Namely, they contain a multiple of the identity on L_1.

Theorem 3.3.2. *Suppose A is a sectorial operator on $L_1(K, \lambda)$. Assume that either:*

(i) A is R-sectorial for some angle ω,

or (ii) $-A$ is generator of a bounded semigroup such that $\{e^{-tA} : 0 < t \leq 1\}$ is R-bounded.

Then there is a bounded function $a(\zeta, s)$ defined for $s \in K$ and $|\arg\zeta| > \omega$ such that

- *For each $s \in K$ the map $\zeta \to a(\zeta, s)$ is analytic.*
- *For each ζ the map $s \to a(\zeta, s)$ is Borel.*
- *$\lambda\{s : \ a(\zeta, s) = 0\} = 0$ for almost every ζ.*
- $$R(\zeta, A)f = a(\zeta, s)f(s) + \int_K f(t)d\mu_s^\zeta(t) \qquad f \in L_1$$

  where $\mu_s^\zeta\{s\} = 0$.

Proof. We begin with the observation that, under either hypothesis, there exists $\phi < \pi$ and $M < \infty$ such that the set of operators $\{\zeta R(\zeta, A) : \ |\arg\phi| \geq \phi, \ |\zeta| \geq M\}$ is R-bounded. Hence again the set of measures $\{\rho_{R(\zeta,A)} : \ |\arg\phi| \geq \phi, \ |\zeta| \geq M\}$ is relatively weakly compact.

Consider the map $\zeta \to \Pi(R(\zeta, A))$ which is an analytic map from the set $\mathcal{S} = \{\zeta : \ |\arg\zeta| > \omega\}$ into $\mathcal{L}(L_1)$. This induces an analytic map $F : \mathcal{S} \to L_\infty(K, \lambda)$ given by

$$\Pi(R(\zeta, A))f = F(\zeta)f.$$

Let us show that we can choose representatives so that $F(\zeta)(s) = a(\zeta, s)$ where a satisfies the first two conditions of the statement. Indeed let $\mathbb{D}$ be the unit disk and let $\varphi : \mathbb{D} \to \mathcal{S}$ be a conformal equivalence. Then $F \circ \phi$ can be expanded in a Taylor series around the origin and we may pick uniformly bounded Borel representatives b_n for the coefficients in the expansion so that

$$F(\varphi(z))(s) = \sum_{n=0}^{\infty} b_n(s) z^n \qquad \lambda - \text{a.e.}.$$

Let

$$a(\zeta, s) = \sum_{n=0}^{\infty} b_n(s)(\varphi^{-1}(\zeta))^n.$$

Assume that the third condition fails. Then by Fubini's theorem there is a subset B of K with $\lambda(B) > 0$ so that for each $s \in B$ the set $\{\zeta : \ a(\zeta, s) = 0\}$ has positive planar measure. By analyticity, this implies $a(\zeta, s) \equiv 0$ for $s \in B$.

However $\rho_{n(n+A)^{-1}}$ converges weakly to ρ_I and hence so does $\rho_{\Pi(n(n+A)^{-1})}$. Thus $-na(-n, s)$ is weakly convergent to the constantly one function in $L_\infty(K, \lambda)$. This is a contradiction. $\square$

The next theorem shows that if a sectorial operator generates an R-bounded semigroup on L_1 then it is very similar to a bounded operator in the sense that its domain is sufficiently large to contain generic L_1-functions.

Theorem 3.3.3. *Let A be a sectorial operator on L_1 and assume that for some $\phi < \pi$ and $M < \infty$ the set $\{\zeta R(\zeta, A) : \ |\arg \zeta| \geq \phi, \ |\zeta| \geq M\}$ is R-bounded. Then for any $\epsilon > 0$ there is an invertible operator $U : L_1 \to L_1$ with $\|U - I\| < \epsilon$ and a density function $w > 0$ a.e. such that $L_1(w) \subset U^{-1}(\mathcal{D}(A))$.*

Proof. According to Proposition 3.2.3 the set of measures $\rho_{\zeta R(\zeta,A)}$ for $|\arg \zeta| \geq \phi, \ |\zeta| \geq M$ is relatively weakly compact in $\mathcal{M}(K \times K)$. The sequence $(n(n + A)^{-1})_{n \geq M}$ converges in the strong operator topology to the identity. Therefore, $\rho_{n(n+A)^{-1}}$ converges *weak** to ρ_I in $\mathcal{M}(K \times K)$ and hence converges weakly to ρ_I by weak compactness.

Now by Proposition 3.2.1 we may find a sequence of convex combinations $(T_n)_{n=1}^{\infty}$ of $(n(n+A)^{-1})_{n=1}^{\infty}$, and a sequence of Borel sets $E_n \subset K$ such that $\lambda(E_n) > 1 - 2^{-n}\epsilon$ and

$$\|T_n f - f\| \leq 2^{-n}\epsilon\|f\| \qquad f \in L_1(E_n).$$

Let us put $F_1 = E_1$ and then $F_n = E_n \setminus E_{n-1}$ for $n \geq 2$. We define $U : L_1 \to L_1$ by

$$Uf = \sum_{n=1}^{\infty} T_n(f\chi_{F_n}).$$

Thus $\|U - I\| \leq \epsilon$. Observe that $T_n : L_1 \to \mathcal{D}(A)$ and so AT_n is a bounded operator on L_1.

Define

$$w = \sum_{n=1}^{\infty} \|AT_n\|\chi_{F_n}$$

and assume $f \in L_1(w)$. Then

$$\|AU(f\chi_{F_n})\| = \|AT_n(f\chi_{F_n})\| \leq \int_{F_n} |f| w \, dt.$$

Hence $\sum_{k=1}^{\infty} AU(f\chi_{F_k})$ converges and, since A is closed, $Uf \in \mathcal{D}(A)$. □

Typically, differential operators on finite domains have compact resolvents. Therefore, they do not generate R-bounded semigroups [28]. In contrast, resolvents of differential operators on infinite domains are generally not compact. An important example is the Laplacian Δ on $L_1(\mathbb{R}^n)$. It does not have a compact resolvent and is hence not covered by the theorem in [28]. Our corollary is addressing this situation.

Corollary 3.3.4. *Let $\Omega \subset \mathbb{R}^n$ with piecewise smooth boundary. Suppose that $A : L_1(\Omega) \supset \mathcal{D}(A) \longrightarrow L_1(\Omega)$ is a sectorial operator such that $\mathcal{D}(A)$ is contained* *in a Sobolev space* $H_1^s(\Omega)$ *for some* $s > 0$*. Then* A *does not generate an R-bounded semigroup.*

Proof. Assume the contrary, i.e. A generates an R-bounded semigroup. Then by Sobolev's theorem we have a continuous inclusion $H_1^s(\Omega) \hookrightarrow L_p(\Omega)$ for some $p > 1$. Using 3.3.3 we obtain that there is a positive weight function w such that $L_1(\Omega, w)$ is isomorphically embedded in $\mathcal{D}(A)$ and thus in $L_p(\Omega)$. This is impossible. $\square$

Chapter 4

The absolute functional calculus

4.1 Maximal regularity

The concept of functional calculus for sectorial operators recently had significant impact on the theory of evolution equations. Namely, the long open maximal regularity problem [36] was solved using functional calculus techniques and Banach space theory [32].

A sectorial operator A on a Banach space X has maximal L_p-regularity if for every $f : \mathbb{R}_+ \longrightarrow X$ there exists a solution $y \in L_p(\mathbb{R}_+, \mathcal{D}(A))$ of the abstract Cauchy equation

$$y'(t) + Ay(t) = f(t)$$

with the additional requirement that $y' \in L_p(\mathbb{R}_+, X)$.

The previous equation can be formally written as $(\tilde{B} + \tilde{A})y = f$ where $\tilde{A}, \tilde{B}$ are the derivative operator and the extension $(\tilde{A}f)(t) = A(f(t))$. Therefore, the maximal regularity problem is equivalent to finding conditions under which $\tilde{B} + \tilde{A}$ is a well-defined closed operator [32]. In this description, a key part of its solution is the following result [32]. The term *R-sectorial* refers to the set $\{\lambda R(\lambda, A)\}$ being R-bounded (recall 1.1.1) outside a sector.

Theorem 4.1.1. *(Kalton and Weis, 2000) Suppose A and B are sectorial operators such that A is H^∞-sectorial and B is R-sectorial. Then $A+B$ is closed on the domain $\mathcal{D}(A)\cap\mathcal{D}(B)$, there is a constant $C>0$ such that*

$$\|Ax\| + \|Bx\| \le C\|Ax + Bx\|, \quad x \in \mathcal{D}(A)\cap\mathcal{D}(B) \tag{4.1}$$

and $(A+B)$ is invertible if either A or B is invertible.

The same conclusion can be obtained if one assumes that X is a UMD space and A, B have so called bounded imaginary powers. This is stronger than R-sectoriality, but weaker than H^∞-calculus. R-sectorial is a stronger notion than merely sectorial. The above theorem however adapts better to applications since differential operators on L_p spaces typically have H^∞-calculus, and L_p has UMD for $1 < p < \infty$.

We are going to give another proof of 4.1.1 while applying our new notion of absolute H^∞-calculus. In doing so we are able to remove any R-boundedness assumption from theorem 4.1.1.

A sectorial operator A has an *absolute H^∞-calculus* if there are a function $g \in H_0^\infty(\Sigma_\phi)$ for some $\phi > \omega$ and $C>0, \delta>0$ such that

$$\|\varphi_{\delta,\delta}(tA)g(tA)x\| \le \|g(tA)y\| \qquad 0<t<\infty \ \Rightarrow \|x\| \le C\|y\|.$$

We are going to discuss the details of this definition later.

4.2 Real interpolation spaces

Absolute H^∞-calculus turns out to be stronger than H^∞-calculus and is extremely powerful when the underlying spaces are real interpolation spaces. We give a far reaching description of spaces on which sectorial operators have absolute calculus. In short, these are spaces obtained by a certain real interpolation method. Our theory is extending the results from [21, 22]. There, it was shown that a sectorial operator has H^∞-calculus on real interpolation spaces between X and $\mathcal{D}(A)$ as well as on real interpolation spaces between X and $\mathcal{D}(A) \cap \mathcal{R}(A)$.

Our main result in this direction says that a sectorial operator A has absolute calculus on any real interpolation space of the form $(\tilde{\mathcal{D}}(A^{-b}), \tilde{\mathcal{D}}(A^a))_F$. This material is discussed in the last section. Moreover, a converse is also presented.

Our framework of absolute calculus is motivated by the existing theory on operators with H^∞-calculus on Lebesgue, Hölder, Sobolev, and Besov spaces. Since these are interpolation spaces, they are naturally covered under our results.

In particular we are able to address the setting of [2, 3] where the maximal regularity problem is formulated for continuous Hölder spaces $C^\alpha(\mathbb{R}), 0 < \alpha < 1$.

We are going to review the above mentioned fundamental notions of sectorial operators and H^∞-calculus together with the construction of joint functional calculus. This will be helpful for the subsequent development of absolute calculus and its applications.

4.3 Sectorial operators

We employ standard notation from Banach space theory. Throughout, X denotes a Banach space, and the letter A is reserved for an unbounded operator on X with domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$. We always suppose that $\mathcal{D}(A)$ and $\mathcal{R}(A)$ are dense. When estimating an expression successively, the actual value of a constant C may change from line to line.

We call

$$\rho(A) = \{\lambda \in \mathbb{C} \,:\, (\lambda I - A) : \mathcal{D}(A) \longrightarrow X \text{ is invertible } \}$$

the resolvent set of A and $\sigma(A) = \mathbb{C} \setminus \sigma(A)$ the spectrum of A. For $\lambda \in \rho(A)$ we can define the resolvent

$$R(\lambda, A) = (\lambda I - A)^{-1} : X \longrightarrow \mathcal{D}(A)$$

as a bounded operator. It is well known that $\rho(A)$ is open hence $\sigma(A)$ closed, and that the map $\lambda \mapsto R(\lambda, A)$ is analytic.

A sector of angle $0 < \omega < \pi$ in the complex plane is defined as

$$\Sigma_\omega = \{\lambda \in \mathbb{C} \,:\, |\arg \lambda| < \omega\} \cup \{0\}$$

The closed operator A is called sectorial of type $0 < \omega < \pi$ if

(i) A is one-to-one.

(ii) $\mathcal{D}(A)$ and $\mathcal{R}(A)$ are dense in X.

(iii) The spectrum $\sigma(A)$ is contained in the sector Σ_ω.

(iv) For every $\omega < \phi < \pi$ there is a constant $C_\phi > 0$ such that for every $\lambda \in \mathbb{C} \setminus \Sigma_\phi$ we have

$$\|\lambda R(\lambda, A)\| \leq C_\phi$$

Notice that this definition does not require A to be invertible.

4.4 Functional calculus for sectorial operators

Denote by $H^\infty(\Sigma_\phi)$ the space of bounded analytic functions on the open sector $\Sigma_\phi \setminus \{0\}$ where $0 < \phi < \pi$. Suppose that A is a sectorial operator of type ω on X and that $\omega < \phi < \pi$. For a function $f \in H^\infty(\Sigma_\phi)$ we can define $f(A)$ as a bounded operator as long as f satisfies a certain growth condition. For $a, b > 0$ define the following utility function.

$$\varphi_{a,b}(z) = \frac{z^a}{(1+z)^{a+b}}$$

Then we consider functions with growth controlled by $\varphi_{a,b}$ and let

$$H_0^\infty(\Sigma_\phi) = \{f \in H^\infty(\Sigma_\phi) \,:\, |f(z)| \leq C\varphi_{\delta,\delta}(|z|) \text{ for some } \delta > 0\ \}$$

Explicitly, this means

$$|f(z)| \leq C\frac{|z|^\delta}{(1+|z|)^{2\delta}}$$

Equivalently, $H_0^\infty(\Sigma_\phi)$ can be seen as the space of all bounded analytic functions on an open sector of angle ϕ such that we have $|f(z)| \leq C\varphi_{a,b}(|z|)$ for some $a, b > 0$ or

$$|f(z)| \leq C|z|^a \text{ for all } |z| \leq 1 \quad \text{and} \quad |f(z)| \leq C|z|^{-b} \text{ for all } |z| > 1$$

We naturally have $\varphi_{a,b} \in H_0^\infty(\Sigma_\phi)$ since $|\varphi_{a,b}(z)| \leq \frac{|z|^a}{(1+|z|^2)^{(a+b)/2}}$.

Now fix $\omega < \phi < \pi$ and a function $f \in H_0^\infty(\Sigma_\phi)$ such that $|f(z)| \leq C\frac{|z|^\delta}{(1+|z|)^{2\delta}}$ for all $z \in \Sigma_\phi$. Suppose that A is sectorial of type ω, and choose any $\omega < \nu < \phi$. Then for any $x \in X$ we can define the contour integral of $f(\zeta)R(\zeta, A)x$ along the path

$$\Gamma = \{|s|e^{i(\text{sgn } s)\nu} \; : \; -\infty < s < \infty\}$$

given by

$$f(A)x = -\frac{1}{2\pi i}\int_\Gamma f(\zeta)R(\zeta, A)x\, d\zeta$$

This is possible due to the parabolic estimate $\|R(\zeta, A)\| \leq C|\zeta|^{-1}$ and the growth conditions on f. Simply notice that

$$\begin{aligned}
\|f(A)x\| &= \|\frac{1}{2\pi}\int_\Gamma f(\zeta)R(\zeta, A)x\, d\zeta\| \\
&\leq \frac{1}{2\pi}\int_\Gamma |f(\zeta)| \cdot \|R(\zeta, A)x\|\, d|\zeta| \\
&\leq \frac{C}{2\pi}\int_{-\infty}^{\infty} |f(|s|e^{i(\text{sgn } s)\nu})|\, \frac{ds}{|s|}\|x\| \\
&\leq C(\int_0^1 s^{\delta-1}\, ds + \int_1^\infty s^{-\delta-1}\, ds)\|x\| \\
&\leq C\|x\|
\end{aligned}$$

Here, the constant C depends on ν. However, by Cauchy's theorem, the element $f(A)x \in X$ does not depend on the choice of ν.

It is shown in [32] that one can extend this definition to an arbitrary bounded analytic function $f \in H^\infty(\Sigma_\phi)$ and all $x \in X$, provided that $\sup_n \|(v_n f)(A)\| < \infty$. Here,

$$v_n(z) = \frac{n}{n+z} - \frac{1}{1+nz}$$

are mollifiers that smooth f into $H_0^\infty(\Sigma_\phi)$. In case $\sup_n \|(v_n f)(A)\| < \infty$ for all $f \in H^\infty(\Sigma_\phi)$ we say that A has H^∞-calculus. Then $f(A)x$ is defined for all $x \in X$ and we have $\|f(A)x\| \leq C\|x\|$.

To address the maximal regularity problem with functional calculus, the Cauchy equation can be formulated as $(A+B)y = f$ for sectorial operators A, B. It is hence of importance to know under which assumptions the sum of two sectorial operators has closed domain $\mathcal{D}(A) \cap \mathcal{D}(B)$.

Following [32] we recall the construction of a joint functional calculus, i.e. obtaining a bounded operator $f(A, B)$ for a bounded analytic function $f(z, w)$.

4.5 Joint functional calculus for sectorial operators

Let A be a sectorial operator of type ω_A on X. We denote by $\mathcal{A}$ the algebra of all bounded operators which commute with $R(\lambda, A)$ for all $\lambda \in \Sigma_{\omega_A}$. We say that two sectorial operators A and B commute if their resolvents $R(\lambda, A)$ and $R(\mu, B)$ commute for all $\lambda \in \Sigma_{\omega_A}$ and $\mu \in \Sigma_{\omega_B}$.

For $\omega < \phi < \pi$ we define $H^\infty(\Sigma_\phi, \mathcal{A})$ as the space of all bounded analytic functions $F : \Sigma_\phi \longrightarrow \mathcal{A}$, so that for every $x \in X$ the map $z \rightarrow F(z)x$ is analytic (i.e. F is analytic for the strong operator topology). We consider the scalar space $H^\infty(\Sigma_\phi)$ embedded as a subspace of $H^\infty(\Sigma_\phi, \mathcal{A})$ via the identification $f \rightarrow fI$. We denote by $H_0^\infty(\Sigma_\phi, \mathcal{A})$ the space of all $F \in H^\infty(\Sigma_\phi, \mathcal{A})$ which obey an estimate of the form $\|F(z)\| \leq C\varphi_{\delta,\delta}(|z|)$ for some $\delta > 0$. Then for any $F \in H_0^\infty(\Sigma_\phi, \mathcal{A})$ and $x \in X$ we define

$$F(A)x = -\frac{1}{2\pi i} \int_\Gamma F(\zeta) R(\zeta, A) x \, d\zeta$$

as a Bochner integral where $\Gamma = \{|s|e^{i(\text{sgn } s)\nu} : -\infty < s < \infty\}$ as in the previously discussed scalar case. Cauchy's theorem shows that this integral does not depend on the choice of ν.

Now pick any $F \in H^{\infty}(\Sigma_{\phi}, \mathcal{A})$ and recall $v_n(z) = \frac{n}{n+z} - \frac{1}{1+nz}$. Then $v_n \in H_0^{\infty}(\Sigma_{\phi})$ and the operators $v_n(A)$ are uniformly bounded with $v_n(A)x \to x$ for every $x \in X$. Since $v_n F \in H_0^{\infty}(\Sigma_{\phi}, \mathcal{A})$, we have that $(v_n F)(A)$ is a bounded operator for every n. It follows from [32] that the extension $F(A)x = \lim_{n\to\infty}(v_n F)(A)x$ defines a bounded operator on X if and only if the operators $(v_n F)(A)$ are uniformly bounded.

Now starting with $f(z, w)$ the technique from [32] applies first scalar valued functional calculus to obtain $f(z, B)$ as a bounded operator for every z and then $f(A, B)$ through operator valued functional calculus. Some conditions on the A, B need to be imposed for taking these steps. This procedure is called the joint functional calculus for A and B.

The development of joint functional calculus is strongly connected to the maximal regularity problem (see [37],[32]). We are looking for the solution y for the equation

$$\begin{cases} Ay + y' = f \\ y(0) = 0 \end{cases} \tag{4.2}$$

where A is a sectorial operator on a Banach space X and $f \in L_p(\mathbb{R}_+, X)$. Then A has *maximal L_p-regularity* if for each $f \in L_p(\mathbb{R}_+, X)$ the Cauchy problem has a unique solution $y \in L_p(\mathbb{R}_+, \mathcal{D}(A))$ such that $y' \in L_p(\mathbb{R}_+, X)$. Suppose $\tilde{B}$ is the derivative operator on $L_p(\mathbb{R}_+, X)$ and $\tilde{A}$ is the extended operator $(\tilde{A}y)(t) = A(y(t))$. Then (4.2) can be written as follows

$$\begin{cases} (\tilde{A} + \tilde{B})y = f \\ y(0) = 0 \end{cases}$$

If the operator $\tilde{A} + \tilde{B}$ with domain $\mathcal{D}(\tilde{A}) \cap \mathcal{D}(\tilde{B})$ has a bounded inverse then we obtain the solution $y = (\tilde{A} + \tilde{B})^{-1} f$. By a result in [16] L_p-maximal regularity is also equivalent to the statement that $\tilde{A} + \tilde{B}$ is closed, and also to an inequality of the form

$$\|\tilde{B}u\| \leq C\|\tilde{A}u\| + \|\tilde{B}u\|$$

for $u \in \mathcal{D}(\tilde{A}) \cap \mathcal{D}(\tilde{B})$. This problem turns out to be closely related to the joint functional calculus. Indeed, $\tilde{A} + \tilde{B}$ being closed is implied by the boundedness of $\tilde{A}(\tilde{A} + \tilde{B})^{-1}$ [36]. For operators with absolute calculus we can improve the result from [32] stated in our introduction (4.1.1). We are now going to focus on the new concept of absolute calculus.

4.6 The definition of absolute functional calculus

Definition 4.6.1. Let A be a sectorial operator of type ω on X. We say that A has an *absolute functional calculus* if there are a function $g \in H_0^\infty(\Sigma_\phi)$ for some $\phi > \omega$ and $C, \delta > 0$ such that

$$\|\varphi_{\delta,\delta}(tA)g(tA)x\| \leq \|g(tA)y\| \qquad 0 < t < \infty \;\Rightarrow \|x\| \leq C\|y\|.$$

Synonymously, we use *absolute H^∞-calculus* or *absolute calculus.*

First, we establish that having absolute functional calculus is stronger than having H^∞-calculus.

Lemma 4.6.2. *Suppose that A is a sectorial operator on X with absolute H^∞-calculus. Then A has H^∞-calculus.*

Proof. Pick any $h \in H^\infty(\Sigma_\phi)$. Notice that $v_n h \in H_0^\infty(\Sigma_\phi)$ and now for any t, n we have

$$\begin{aligned}\|\varphi_{\delta,\delta}(tA)(v_n h)(A)\| &\le \int_\Gamma \|\varphi_{\delta,\delta}(t\zeta)v_n(\zeta)h(\zeta)R(\zeta,A)\|\, d|\zeta| \\ &\le C\int_\Gamma |\varphi_{\delta,\delta}(t\zeta)h(\zeta)|\frac{d|\zeta|}{|\zeta|} \\ &\le \max_{\zeta\in\Sigma}|h(\zeta)|\int_\Gamma |\varphi_{\delta,\delta}(t\zeta)|\frac{d|\zeta|}{|\zeta|} \\ &\le C\|h\|_\infty\end{aligned}$$

Denote $M = \sup_{t,n}\|\varphi_{\delta,\delta}(tA)(v_n h)(A)\|$. We take $y \in X$ and $x_n = \frac{1}{M}(v_n h)(A)y$. Pick $g \in H_0^\infty(\Sigma_\omega)$ from the definition of absolute calculus. We have

$$\begin{aligned}\|\varphi_{\delta,\delta}(tA)g(tA)x_n\| &= \|\varphi_{\delta,\delta}(tA)g(tA)\frac{1}{M}(v_n h)(A)y\| \\ &\le \frac{1}{M}\|\varphi_{\delta,\delta}(tA)(v_n h)(A)\|\cdot\|g(tA)y\| \\ &\le \|g(tA)y\|\end{aligned}$$

Therefore, by the definition of absolute H^∞-calculus we get $\frac{1}{M}\|(v_n h)(A)y\| = \|x_n\| \le C\|y\|$. Hence, $\|(v_n h)(A)y\| \le CM\|y\|$ for all y contained in a dense subset of X. This shows that $(v_n h)(A)$ extends to a sequence of bounded operators which is uniformly bounded and thus A has H^∞-calculus. □

Sectorial operators with H^∞-calculus on Hilbert spaces, L_1- and $C(K)$-spaces already have absolute calculus. This fact explains from a different point of view the results for R-bounded semigroups on L_1 and $C(K)$ which were obtained in [28]. Even more is true in the setting of GT-spaces. Recall that a Banach space X is a GT-space if every bounded operator $T : X \longrightarrow \ell_2$ is absolutely summing, i.e. there exists $K > 0$ such that

$$\sum_{k=1}^{n} \|Tx_k\| \leq K \max_{\alpha_k=\pm 1} \|\sum_{k=1}^{n} \alpha_k x_k\|$$

for every collection $x_1, \ldots, x_n \in X$. Important examples of GT-spaces are L_1 and L_1/H_1 (see [12], [13], and [45]). Examples of spaces for which the dual is a GT-space are $C(K)$ and the disc algebra $A(\mathbb{D})$.

Theorem 4.6.3. *Let A be a sectorial operator with H^∞-calculus on X. Suppose one of the following holds*

(i) X is a Hilbert space

(ii) X is a GT-space

(iii) X^* is a GT-space

Then A has absolute calculus on X.

Proof. Suppose that a sectorial operator A has H^∞-calculus on a Hilbert space X. Then the first statement follows from the results of M^{c}Intosh [39],[4]. He showed that in this case the quadratic norm is equivalent to the given norm on X:

$$\|x\| \approx (\int_0^\infty \|\psi(tA)x\|^2 \frac{dt}{t})^{1/2}.$$

where $\varphi(z)$ is any bounded holomorphic function in the sector such that

$$|\psi(z)| \leq C|z|^s(1 + |z|^{2s})^{-1}$$

for some constant $C > 0$ and $s > 0$.

Now suppose that X is a GT-space and A is a sectorial operator of type ω with H^∞-calculus. We will show that if $f(z) \in H_0^\infty(\Sigma_\phi)$ for $\omega < \phi < \pi$ then there is a constant $C > 0$ so that

$$C^{-1}\|x\| \leq \int_0^\infty \|f(tA)x\|\frac{dt}{t} \leq C\|x\|. \tag{4.3}$$

We rewrite $\int_0^\infty \|f(tA)x\|\frac{dt}{t} = \int_1^2 \sum_{k\in\mathbb{Z}} \|f(2^k tA)x\|\frac{dt}{t}$ and fix $x \in X$ and $t > 0$. By the Hahn-Banach theorem choose functionals $x_k^* \in X^*$ with $\|x_k^*\| = 1$ and $\langle f(2^k tA)x, x_k^*\rangle = \|f(2^k tA)x\|$.

Let $g(z) = f(z)^{\frac{1}{2}}$. Since A has H^∞-calculus there is a constant $C_0 > 0$ independent of x and t such that if $\alpha_k = \pm 1$ then

$$\|\sum_{|k|<n} \alpha_k g(2^k tA)x\| \leq C_0\|x\|.$$

This can be found in [32, chapter 4]. Any GT-space has the Orlicz property [45, chapter 6c], i.e. there is a constant $C_1 > 0$ such that for any finite sequence (x_j) in X we have

$$(\sum_j \|x_j\|^2)^{\frac{1}{2}} \leq \sup_{\alpha_j=\pm 1} \|\sum_j \alpha_j x_j\|$$

Therefore, we obtain

$$(\sum_{k\in\mathbb{Z}} \|g(2^k tA)x\|^2)^{\frac{1}{2}} \leq C_0 C_1\|x\|.$$

Consequently, the operator $S_x; X \longrightarrow \ell_2(\mathbb{Z}), S_x y = (\langle g(2^k tA)y, x_k^*\rangle)_{k\in\mathbb{Z}}$ is bounded and thus absolutely summing. We hence obtain

$$\begin{aligned}\sum_{|k|\le N} \|f(2^k tA)x\| &= \sum_{|k|\le N} \langle f(2^k tA)x, x_k^*\rangle \\ &\le \sum_{|k|\le N} \|S_x g(2^k tA)x\| \\ &\le K \max_{\alpha_k=\pm 1} \|\sum_{|k|\le N} \alpha_k g(2^k tA)x\| \\ &\le KC_0\|x\|\end{aligned}$$

Integrating over the interval $[1,2]$ yields the upper estimate of (4.3).

For the lower estimate note that $\int_0^\infty f(tz)\frac{dt}{t}$ does not depend on z. Therefore $\int_0^\infty f(tA)\frac{dt}{t} = cI$ where I is the identity operator and $c = \int_0^\infty f(tz)\frac{dt}{t}$. Now for every $x \in X$ we have

$$\int_0^\infty \|f(tA)x\|\frac{dt}{t} \ge \|\int_0^\infty f(tA)\frac{dt}{t}x\| = c\|x\|.$$

To show (iii) suppose that X^* is a GT-space. Pick $f \in H_0^\infty(\Sigma_\phi)$. Then we also have $f^2 \in H_0^\infty(\Sigma_\phi)$ and the argument from above gives

$$C^{-1}\|x^*\| \le \int_0^\infty \|f^2(tA)^*x^*\|\frac{dt}{t} \le C\|x^*\|, \quad x^* \in X^*.$$

We are going to derive

$$c^{-1}\|x\| \le \sup_{t>0} \|f(tA)x\| \le c\|x\|.$$

Let $d = \int_0^\infty f^2(tz)\,\frac{dt}{t}$. The upper estimate holds since $f(tA)$ are uniformly bounded. Choose a norming functional x^* so that $\|x^*\| = 1$ and $\langle x^*, x\rangle = \|x\|$. Then for the lower estimate we have

$$\begin{aligned}
\|x\| &= \langle x, x^*\rangle \\
&= d^{-1}\int_0^\infty \langle f^2(tA)x, x^*\rangle \,\frac{dt}{t} \\
&\leq d^{-1}\int_0^\infty \langle f(tA)x, f(tA)^*x^*\rangle \,\frac{dt}{t} \\
&\leq d^{-1}\sup_{t>0}\|f(tA)x\| \cdot \int_0^\infty \|f(tA)^*x^*\| \,\frac{dt}{t} \\
&\leq d^{-1}C\sup_{t>0}\|f(tA)x\|
\end{aligned}$$

This concludes the theorem.

□

Choose any $f \in H_0^\infty(\Sigma_\omega), x \in X$, and $t > 0$. As seen before, we get an estimate

$$\|f(tA)x\| \leq C\int_{-\infty}^{\infty} |f(|s|e^{i(\text{sgn } s)\nu}| \,\frac{ds}{|s|} \tag{4.4}$$

We are now considering properties of the function

$$t \mapsto \|f(tA)x\|_X$$

This function is the basis of the associated norm $\|\cdot\|_{X(f)}$ discussed later on.

We start with a lemma regarding growth restrictions of $u \mapsto \|f(utA)g(tA)x\|$.

Lemma 4.6.4. *Suppose $a, a', b, b' > 0$ and $f, g \in H^\infty(\Sigma_\phi)$ where $\phi > \omega$. Suppose that f, g satisfy $|f(z)| \leq C\varphi_{a,b}(|z|)$ and $|g(z)| \leq C\varphi_{a',b'}(|z|)$, i.e. we have estimates*

$$|f(z)| \leq C_0|z|^a(1+|z|)^{-(a+b)}$$

$$|g(z)| \leq C_0|z|^{a'}(1+|z|)^{-(a'+b')}.$$

Then there exists a constant $C = C(a, b, f, g)$ so that

$$\|f(utA)g(tA)\| \leq Cu^{\min(a,b')}(1+u)^{-\min(a,b')-\min(a',b)}(1+|\log u|) \qquad u > 0, t > 0.$$

Proof. By the choice of f, g we have

$$|f(uz)||g(z)| \le Cu^a|z|^{a+a'}(1+u|z|)^{-(a+b)}(1+|z|)^{-(a'+b')}.$$

By (4.4) we have

$$\begin{aligned}\|f(utA)g(tA)\| &\le \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} |f(use^{i\text{sgn }(s)})g(se^{i\text{sgn }(s)})|\,\frac{ds}{s} \\ &\le Cu^a\int\limits_{0}^{\infty} s^{a+a'}(1+us)^{-(a+b)}(1+s)^{-(a'+b')}\,\frac{ds}{s}\end{aligned}$$

Suppose $u \ge 1$. Then since $(1+us)^{-(a+b)}(1+s)^{-(a'+b')} < 1$ we have

$$\int_0^{u^{-1}} s^{a+a'-1}(1+us)^{-(a+b)}(1+s)^{-(a'+b')}ds \le \int_0^{u^{-1}} s^{a+a'-1}ds = \frac{1}{a+a'}u^{-a-a'}$$

Since $(1+us)^{-(a+b)} \le (us)^{-(a+b)}$ we get

$$\begin{aligned}\int_{u^{-1}}^{1} s^{a+a'-1}(1+us)^{-(a+b)}(1+s)^{-(a'+b')}ds &\le u^{-(a+b)}\int_{u^{-1}}^{1} s^{a'-b-1}ds \\ &= \begin{cases}\frac{1}{a'-b}(u^{-a-b}-u^{-a-a'}) \text{ if } a' \ne b\\ u^{-a-b}\ln\, u \text{ if } a' = b\end{cases}\end{aligned}$$

For the last integral we also use that $(1+s)^{-(a'+b')} < s^{-(a'+b')}$

$$\int_1^{\infty} s^{a+a'-1}(1+us)^{-(a+b)}(1+s)^{-(a'+b')}ds \le u^{-(a+b)}\int_1^{\infty} s^{-1-b-b'}ds = \frac{1}{b+b'}u^{-a-b}$$

Since $u > 1$ we have $u^{-a'} \le u^{-min(a',b)}$ and $u^{-b} \le u^{-min(a',b)}$. Therefore for a suitable constant $C > 0$ we obtain

$$\|f(utA)g(tA)\| \le Cu^a u^{-a-min(a',b)}(1+|\ln\, u|) = Cu^{-min(a',b)}(1+|\ln\, u|)$$

If $u \le 1$ we observe that $f(utA)g(tA) = f(sA)g(u^{-1}sA)$ where $s = ut$. In this case the previous estimate gives the following

$$\|f(utA)g(tA)\| \le Cu^{min(a,b')}(1+|\ln\, u|)$$

Combining these two estimates we obtain

$$\|f(utA)g(tA)\| \leq Cu^{\min(a,b')}(1+u)^{-\min(a,b')-\min(a',b)}(1+|\log u|).$$

That gives us the result. $\square$

The next theorem states that absolute calculus implies operator functional calculus. Its proof uses the previous lemma. This theorem implies the scalar case as well which we had already established. It is is fact a stronger version of theorem 4.4 from [32].

Theorem 4.6.5. *Suppose A is a sectorial operator of type ω with absolute calculus and $F \in H^\infty(\Sigma_\phi, \mathcal{A})$ for some $\phi > \omega$. Then $F(A)$ is a bounded operator on X.*

Proof. Since A has absolute calculus, there is a function $g \in H_0^\infty(\Sigma_\phi)$, $\phi > \omega$ and a positive number δ such that

$$\|\varphi_{\delta,\delta}(tA)g(tA)x\| \leq \|g(tA)y\| \qquad 0 < t < \infty \;\Rightarrow \|x\| \leq C\|y\|.$$

We fix $\omega < \nu < \phi$ and $0 < \gamma < 1$. It follows from the result in [32] that for any $F_0 \in H_0^\infty(\Sigma_\phi, \mathcal{A})$ we have the representation

$$F_0(A)x = -\frac{1}{2\pi i}\int_{\Gamma_\nu} \zeta^{-\gamma}F_0(\zeta)A^\gamma R(\zeta, A)x\,d\zeta$$

Using the parametrization $\zeta = |s|e^{i(\operatorname{sgn} s)\nu}$ for $-\infty < s < \infty$ and functions $h^\rho(z) = z^\gamma(e^{i\rho} - z)^{-1}$ we can rewrite

$$\begin{aligned}F_0(A)x = &\frac{e^{i(1-\gamma)\nu}}{2\pi i}\int_0^\infty F_0(se^{i\nu})h^\nu(s^{-1}A)x\,\frac{ds}{s}\\ &+\frac{e^{-i(1-\gamma)\nu}}{2\pi i}\int_0^\infty F_0(se^{-i\nu})h^{-\nu}(s^{-1}A)x\,\frac{ds}{s}\end{aligned}$$

Denote $y = \int_0^\infty F_0(se^{i\nu}) h^\nu(s^{-1}A) x \, \frac{ds}{s}$. Then for every $t > 0$ we have

$$\begin{aligned}
\|\varphi_{\delta,\delta}(tA) g(tA) y\| &\le \int\limits_0^\infty \|F_0(se^{i\nu})\varphi_{\delta,\delta}(tA) h^\nu(s^{-1}A) g(tA) x\| \, \frac{ds}{s} \\
&\le \sup_{z\in\Sigma_\phi} \|F_0(z)\| \, \|g(tA)x\| \int\limits_0^\infty \|\varphi_{\delta,\delta}(tA) h^\nu(s^{-1}A)\| \, \frac{ds}{s}
\end{aligned}$$

Now we will estimate $\|\varphi_{\delta,\delta}(tA) h^\nu(s^{-1}A)\|$. For $\omega < \rho < \nu$ we have that $h^\nu(z) \in H_0^\infty(\Sigma_\rho)$ and satisfy the estimate $|h^\nu(z)| \le C_\rho |z|^\gamma (1+|z|)^{-1}$. Therefore by Lemma 4.6.4 there exists a constant $C > 0$ so that

$$\begin{aligned}
\|\varphi_{\delta,\delta}(tA) h^\nu(s^{-1}A)\| &= \|\varphi_{\delta,\delta}(tA) h^\nu(t(ts)^{-1}A)\| \\
&\le C \frac{(ts)^{-\min(\gamma,\delta)}}{(1+(ts)^{-1})^{\min(\gamma,\delta)+\min(1-\gamma,\delta)}} (1 + |\log(ts)|) \\
&= C \frac{(ts)^{\min(1-\gamma,\delta)}}{(1+ts)^{\min(\gamma,\delta)+\min(1-\gamma,\delta)}} (1 + |\log(ts)|)
\end{aligned}$$

Since $\int_0^\infty \frac{(ts)^{\min(1-\gamma,\delta)}}{(1+ts)^{\min(\gamma,\delta)+\min(1-\gamma,\delta)}} (1 + |\log(ts)|) \, \frac{ds}{s}$ converges and does not depend on t we obtain

$$\|\varphi_{\delta,\delta}(tA) g(tA) y\| \le C \sup_{z\in\Sigma_\phi} \|F_0(z)\| \, \|g(tA)x\|, \quad 0 < t < \infty.$$

Since A has absolute calculus we conclude that

$$\|y\| \le C \sup_{z\in\Sigma_\phi} \|F_0(z)\| \, \|x\|.$$

The second integral in the representation of $F_0(A)$ can be estimated in exactly the same way. Therefore we obtain

$$\|F_0(A)x\| \le C \sup_{z\in\Sigma_\phi} \|F_0(z)\| \, \|x\|.$$

Thus for the functions $v_n(z)$ defined above we have that there exists a constant $C > 0$ such that

$$\|v_n F(A)x\| \le C \sup_{z\in\Sigma_\phi} \|v_n F(z)\| \, \|x\|.$$

Therefore operators $v_n F(A)$ are uniformly bounded and hence we can define $F(A)$ as a bounded operator. □

Let us apply this theorem to the case of two operators. It is important to note that we do not require any R-boundedness in regards to A and B.

Corollary 4.6.6. *Suppose A and B are commuting sectorial operators of types ω_A and ω_B and A has absolute calculus. Suppose further that $f \in H^\infty(\Sigma_\phi \times \Sigma_\rho)$ where $\phi > \omega_A$, $\rho > \omega_B$ is such that $f(z, B)$ is a bounded operator for every $z \in \Sigma_\phi$ and the family $\{f(z, B) : z \in \Sigma_\phi\}$ is uniformly bounded. Then $f(A, B)$ is a bounded operator.*

Proof. Consider $F(z) = f(z, B)$. Using the integral representation of $f(z, B)$ from section 2 and commutativity of A and B we conclude that $F(z) \in H^\infty(\Sigma_\phi, \mathcal{A})$. It follows from Theorem 4.6.5 that $F(A)$ is a bounded operator. □

For operators with absolute calculus we can improve a key result in [32]. Again, the crucial difference is that we do not need any operator to be R-sectorial.

Theorem 4.6.7. *Let A and B are commuting sectorial operators of types ω_A and ω_B so that $\omega_A + \omega_B < \pi$. Suppose A has absolute calculus. Then $A + B$ is closed on the domain $\mathcal{D}(A) \cap \mathcal{D}(B)$, there is a constant C such that*

$$\|Ax\| + \|Bx\| \leq C\|Ax + Bx\|, \quad x \in \mathcal{D}(A) \cap \mathcal{D}(B) \tag{4.5}$$

and $(A + B)$ is invertible if either A or B is invertible.

Proof. Choose ϕ, ϕ' with $\omega(A) < \phi, \omega(B) < \phi'$ and $\phi + \phi' < \pi$. The function $f(z, w) = z(z + w)^{-1}$ is in $H^\infty(\Sigma_\phi \times \Sigma'_\phi)$ and the family $f(z, B) = -zR(-z, B)$

for $z \in \Sigma'_\phi$ is uniformly bounded. By Corollary 4.6.6 we obtain that $f(A,B)$ is a bounded operator. It was shown in [36] and [32] that this implies the norm estimate 4.5. For completeness we will include the argument from [32]. Using the functions $v_n(z)$ defined above note that $zv_n(z)^2 \in H^\infty(\Sigma_{\phi+\phi'})$. Thus $Av_n(A)^2$ and $Bv_n(B)^2$ are bounded operators. Now for $x \in \mathcal{D}(A) \cap \mathcal{D}(B)$ we have

$$f(A,B)(A+B)v_n(A)^2v_n(B)^2x = v_n(A)^2v(B)^2Ax.$$

Therefore

$$\|v_n(A)^2v(B)^2Ax\| \leq C\|v_n(A)^2v_n(B)^2(A+B)x\|$$

where $C = \|f(A,B)\|$. Letting $n \to \infty$ yields the result.

From the estimate $\|Ax\| + \|Bx\| \leq C\|(A+B)x\|$ we obtain that $A+B$ is closed as follows. Take $x_n \in \mathcal{D}(A) \cap \mathcal{D}(B), x, y \in X$ such that $x_n \longrightarrow x$ and $(A+B)x_n \longrightarrow y$. Since $\|(A+B)(x_n - x_m)\| \longrightarrow 0$ we have $A(x_n - x_m) \longrightarrow 0$ and $B(x_n - x_m) \longrightarrow 0$. A and B are closed and therefore $x \in \mathcal{D}(A) \cap \mathcal{D}(B)$ with $Ax_n \longrightarrow Ax$ and $Bx_n \longrightarrow Bx$. Now $Ax_n + Bx_n \longrightarrow Ax + Bx = y$.

Now suppose that A is invertible. Then since $A(A+B)^{-1}$ is bounded we conclude that also $(A+B)$ is invertible. In the case B is invertible we write $I = A(A+B)^{-1} + B(A+B)^{-1}$ and see that $B(A+B)^{-1}$ is bounded and thus $(A+B)$ invertible. □

4.7 Köthe function spaces

There is a natural way of defining a new norm on X associated to a triple (A, f, E) where A is a sectorial operator, E is a Köthe function space, and $f \in H_0^\infty(\Sigma_\phi)$. Before we introduce this norm we review important facts about Köthe function spaces, especially the behavior of dilations and Boyd indices.

Consider the measure space $(0, \infty)$ with measure dt/t. Let $L_0(0, \infty)$ denote the space of all measurable functions on $(0, \infty)$ where functions coinciding almost everywhere are identified. We denote by $D_s : L_0 \to L_0$ the linear map $D_s f(t) = f(t/s)$. Let E be a linear subspace of L_0 equipped with a norm $\| \cdot \|_E$ so that E is a Banach space. We shall say that E is an *admissible Banach function space* if the following conditions hold:

(i) E is a Köthe function space, i.e. E consists of locally integrable real valued functions, contains characteristic functions of bounded intervals and if f, g are measurable functions with $|f| \leq |g|$ a.e. and $g \in E$ then $f \in E$ and $\|f\|_E \leq \|g\|_E$.

(ii) E has the Fatou property i.e. if (f_n) is an increasing sequence of non-negative functions in E such that $\sup_n \|f_n\|_E < \infty$ then $\sup_n f_n \in E$ and $\| \sup_n f_n\|_E = \sup_n \|f_n\|_E$.

(iii) For each $s \in (0, \infty)$ the operator $D_s : E \to E$ is bounded.

Every Köthe function space is a Banach lattice in the obvious order ($f \geq 0$ if $f(t) \geq 0$ a.e.). This lattice is $\sigma-$order complete. Indeed, if $\{f_n\}_{n=1}^\infty$ is an order bounded increasing sequence in E then $f(t) = \lim_{n\to\infty} f_n(t)$ is the l.u.b. of $\{f_n\}_{n=1}^{\infty}$.

Let E' be the Köthe dual, i.e. E' is the set of $f \in L_0(0,\infty)$ such that

$$\|f\|_{E'} = \sup_{\|g\|_E \leq 1} \int_0^{\infty} |f(t)||g(t)| \frac{dt}{t} < \infty. \tag{4.6}$$

Since E has Fatou property E' is norming for E and is itself admissible [38] pp. 29; furthermore $E'' = E$.

For an admissible space E *Boyd indices* are defined as

$$\beta_E = \limsup_{s\to\infty} \frac{\log \|D_s\|}{\log s}$$

$$\alpha_E = \liminf_{s\to 0} \frac{\log \|D_s\|}{\log s}$$

Lemma 4.7.1. *Let E be an admissible Banach Function space. Then*

1) $-\infty < \alpha_E \leq \beta_E < \infty$

2) If $a < \alpha_E$ and $b > \beta_E$ then there exists a constant $C = C(E, a, b)$ such that

$$\|D_s\| \leq C s^a (1+s)^{b-a}$$

3) $\|D_s\|_E = \|D_{\frac{1}{s}}\|_{E'}$

Proof. First note that the map $s \to \|D_s\|$ is lower semi-continuous and hence measurable. Since the set $V_M = \{s : \|D_s\|, \|D_{s^{-1}}\| \leq M\}$ is measurable and $(0,\infty) = \cup_M V_M$ we can find M so that the set V_M has positive measure. Therefore $V_M V_M^{-1}$ contains a neighborhood of 1 and hence there exists $c > 1$ so that $\|D_s\| \leq M^2$ if $c^{-1} < s < c$. For $s > c$ pick a natural number n_s so that $\frac{\log s}{\log c} < n_s < \frac{\log s}{\log c} + 1$. Then $1 < s^{1/n_s} < c$. From the submultiplicativity of $s \to \|D_s\|$ we obtain

$$\log \|D_s\| = \log \|D_{s^{1/n_s}}^{n_s}\| \leq n_s \log \|D_{s^{1/n_s}}\| \leq (\frac{\log s}{\log c} + 1) \log M^2.$$

Therefore $\beta_E \leq \limsup_{s\to\infty}(\frac{\log M^2}{\log c} + \frac{logM^2}{\log s}) \leq \frac{\log M^2}{\log c} < \infty$.

In the same way we obtain $\alpha_E \geq \liminf_{s\to 0}(\frac{logM^2}{\log s} - \frac{\log M^2}{\log c}) \geq -\frac{\log M^2}{\log c} > -\infty$.

Since $\|D_s\|\,\|D_{\frac{1}{s}}\| \geq \|D_s D_{\frac{1}{s}}\| = 1$ we have $\log\|D_s\| + \log\|D_{\frac{1}{s}}\| \geq 0$. Thus for $0 < s < 1$ we obtain

$$\beta_E = \limsup_{s\to 0} \frac{\log\|D_{\frac{1}{s}}\|}{\log\frac{1}{s}} \geq \limsup_{s\to 0} \frac{-\log\|D_s\|}{-\log s} \geq \liminf_{s\to 0} \frac{\log\|D_s\|}{\log s} = \alpha_E.$$

This ends the proof of the first statement.

By the definition of α_E and β_E there are s_0 and s_1 such that for all $0 < s \leq s_0$ we have $\|D_s\| \leq s^a$ and for all $s > s_1$ we have $\|D_s\| \leq s^b$.

Notice that by lower semi-continuity there is a constant $C > 0$ such that $\|D_s\| \leq C(s^a + s^b)$ for all $s_0 \leq s \leq s_1$. Then we obtain for any $s > 0$ that

$$\begin{aligned} \|D_s\| &\leq C(s^a + s^b) + s^a + s^b \\ &\leq (C+1)(s^a + s^b) \\ &= (C+1)s^b(1 + s^{b-a}) \\ &\leq 2(C+1)s^b(1+s)^{b-a} \end{aligned}$$

Finally, the third statement can be obtained by change of variables in (4.6). $\square$

Remark 4.7.2. If E is rearrangement invariant Banach lattice then $s \to \|D_s\|$ is a nondecreasing function [38] and $0 \leq \alpha_E \leq \beta_E \leq 1$.

We now give a few examples to illustrate our setup.

(i) If $E = L_p(dt)$ we have $\|D_s f\| = (\int_0^\infty |f(t/s)|^p dt)^{1/p} = s^{1/p}\|f\|$ and so $\|D_S\| = s^{1/p}$. Hence, $\alpha_{L_p} = \beta_{L_p} = 1/p$.

(ii) If $E = L_p(dt/t)$ then $\|D_s f\| = (\int_0^\infty |f(t/s)|^p dt/t)^{1/p} = \|f\|$ and $\|D_s f\| = 1$. So in this case we have $\alpha_E = \beta_E = 0$.

(iii) If $E = L_p(t^{-\theta p} dt/t)$ for $\theta \in (0,1)$ then $\|D_s f\| = (\int_0^\infty |f(t/s)|^p t^{-\theta p} \frac{dt}{t})^{1/p} = s^{-\theta}\|f\|$ and $\|D_s f\| = s^{-\theta}$. So in this case we have $\alpha_E = \beta_E = -\theta$.

4.8 The associated spaces $X_E(f)$

Choose any $f \in H_0^\infty(\Sigma_\omega), x \in X$, and $t > 0$. As seen before, we get an estimate

$$\|f(tA)x\| \leq C \int_{-\infty}^{\infty} |f(|s|e^{i(\operatorname{sgn} s)\nu})| \frac{ds}{|s|}$$

We are now considering the function

$$t \mapsto \|f(tA)x\|_X$$

as a possible element of E and define a new norm on X by

$$\|x\|_{X_E(f)} = \begin{cases} \|\|f(tA)x\|_X\|_E & \text{if } \|f(tA)x\|_X \in E \\ \infty \quad \text{otherwise} \end{cases}$$

It is interesting to note that for our utility function $\varphi_{a,b}(z) = \frac{z^a}{(1+z)^{a+b}}$, which was used in the definition of $H_0^\infty(\Sigma_\omega)$, the norm constructed from $\varphi_{a,b}$ is equivalent to the norm constructed from

$$\psi_{a,b}(z) = \frac{z^a}{1 + z^{a+b}}$$

This is the contents of the next lemma.

Lemma 4.8.1. *Let A be a sectorial operator and $\psi_{a,b}(z) = \frac{z^a}{1+z^{a+b}}$ for some $a, b > 0$. Then there is a constant C such that for every $t > 0$ and $x \in X$ we have*

$$C^{-1}\|\psi_{a,b}(tA)x\| \leq \|\varphi_{a,b}(tA)x\| \leq C\|\psi_{a,b}(tA)x\|$$

Proof. We show that operators $\frac{\varphi}{\psi}(tA)$ are uniformly bounded. Suppose $|z| \leq 1/2$. Then

$$\frac{\varphi(z)}{\psi(z)} - 1 = \frac{1 + z^{a+b} - (1+z)^{a+b}}{(1+z)^{a+b}} = \frac{z^{\min(1,a+b)} f(z)}{(1+z)^{a+b}}$$

where $f(z)$ is a bounded function on $|z| \leq 1/2$. Thus

$$\left|\frac{\varphi(z)}{\psi(z)} - 1\right| \leq \frac{|z|^{\min(1,a+b)}}{(1/2)^{a+b}} \sup_{|z|\leq 1/2} |f(z)| = C_1 |z|^{\min(1,a+b)}$$

It follows that for $|z| \geq 2$ we have

$$\left|\frac{\varphi(z)}{\psi(z)} - 1\right| = \left|\frac{\varphi(1/z)}{\psi(1/z)} - 1\right| \leq C_1 \frac{1}{|z|^{\min(1,a+b)}}$$

Now using $\|f(tA)x\| \leq C \int_{-\infty}^{\infty} |f(|s|e^{i(\text{sgn } s)\nu})| \frac{ds}{|s|}$ for every $t > 0$ we obtain

$$\|(\frac{\varphi}{\psi}(tA) - I)x\| \leq \frac{C_\phi}{2\pi} \int\limits_{-\infty}^{\infty} \left|\frac{\varphi}{\psi}(se^{i(\text{sgn } s)\nu}) - 1\right| \frac{ds}{|s|}$$

The last integral splits into three parts, namely over the domains $\{|s| \leq 1/2\}$, $\{1/2 < |s| \leq 2\}$, and $\{|s| > 2\}$. Using the previous estimates we see that these integrals are bounded. Therefore, the operators $\frac{\phi}{\psi}(tA) - I$ and hence $\frac{\phi}{\psi}(tA)$ are uniformly bounded. Then for every x,

$$\|\varphi(tA)x\| = \|\frac{\varphi\psi}{\psi}(tA)x\| \leq \|\frac{\varphi}{\psi}(tA)\| \cdot \|\psi(tA)x\| \leq C\|\psi(tA)x\|$$

In the same way we can obtain the opposite inequality. □

Definition 4.8.2. Suppose E is an admissible Banach function space and $f \in H_0^\infty(\phi)$. Consider the linear subspace of X

$$\tilde{X}_E(f) = \{x \in X : \|x\|_{X_E} < \infty \text{ and } \lim_{s\to 0} \|s(s+A)^{-1}x\|_{X_E(f)} = \lim_{s\to\infty} \|A(s+A)^{-1}x\|_{X_E(f)} = 0\}$$

We define $X_E(f)$ as the completion of $\tilde{X}_E(f)$ under the norm $\|\cdot\|_{X_E(f)}$.

It turns out that sectorial operators always have absolute functional calculus on $X_E(f)$ for any admissible space E and $f \in H_0^\infty(\Sigma_\phi)$. Moreover, there are actually no other spaces where absolute calculus is present.

We now show that sectoriality transfers to $X_E(f)$.

Lemma 4.8.3. *Suppose A is a sectorial operator of type ω on a Banach space X. Then A is a sectorial operator of type ω on $X_E(f)$ for any admissible space E and function $f \in H_0^\infty(\Sigma)$.*

Proof. First note that if T is a bounded operator on X that commute with $f(tA)$ for all t then $\|Tx\|_{X_E(f)} = \|\|f(tA)Tx\|_X\|_E \leq \|T\|_X\|\|f(tA)x\|_X\|_E \leq \|T\|_X\|x\|_{X_E(f)}$ and thus T is bounded on $X_E(f)$ with the norm estimate $\|T\|_{X_E(f)} \leq \|T\|_X$.

We consider an operator A on $X_E(f)$ with the domain

$$\mathcal{D}_{X_E(f)}(A) = \{x \in X_E(f) : \|Ax\|_{X_E(f)} < \infty\}.$$

For every $s > 0$ the operator $A(s+A)^{-1} = -AR(-s, A)$ is bounded on X and thus on X_E. Since $\|A(s+A)^{-1}x\|_{X_E(f)} < \infty$ elements $s(s+A)^{-1}x$ belong to $\mathcal{D}_{X_E(f)}(A)$ for all $x \in X_E$ and $s > 0$. Take any $x \in X_E(f)$. Then

$$\lim_{s\to\infty} \|x - s(s+A)^{-1}x\|_{X_E(f)} = \lim_{s\to\infty} \|A(s+A)^{-1}x\|_{X_E(f)} = 0$$

Therefore $\mathcal{D}_{X_E(f)}(A)$ is dense in $X_E(f)$.

To show that the range of A is dense in $X_E(f)$ we look at the elements $A(s+a)^{-1}x$. Since for every $x \in X_E(f)$ we have $A(s+a)^{-1}x \in \mathcal{R}_{X_E(f)}(A)$ end

$$\lim_{s\to\infty} \|x - A(s+A)^{-1}x\|_{X_E(f)} = \lim_{s\to\infty} \|s(s+A)^{-1}x\|_{X_E(f)} = 0,$$

we conclude that $\mathcal{R}_{X_E(f)}(A)$ is dense in $X_E(f)$.

The resolvent estimate follows from our first observation, i.e.

$$\|\lambda R(\lambda, A)\|_{X_E(f)} \leq \|\lambda R(\lambda, A)\|_X \leq C_\phi, \text{ with } |\arg \lambda| > \phi > \omega.$$

$\square$

Remark 4.8.4. If A is a sectorial operator on X, then for any $x \in \mathcal{D}(A)$ we have

$$\lim_{s \to \infty} \|(s+A)^{-1}Ax\| \leq \lim_{s \to \infty} \frac{C}{s}\|Ax\| = 0,$$

while if $x = Ay$ is in the range of A we get

$$\lim_{s \to 0} \|s(s+A)^{-1}x\| = \lim_{s \to 0} \|s(s+A)^{-1}Ay\| \leq \lim_{s \to 0} Cs\|y\| = 0.$$

Since both the domain and range are dense this means that the above limits are zero for all $x \in X$.

The next theorem uses a lemma interesting on its own.

Lemma 4.8.5. *Let X be a Banach space and $x(t) : [c,d] \mapsto X$ be a continuous function such that $x(t) \neq 0$ for all $t \in [c,d]$. Suppose that $x_c^*, x_d^* \in B_{X^*}$ satisfy $\langle x_c^*, x(c)\rangle = \|x(c)\|$ and $\langle x_d^*, x(d)\rangle = \|x(c)\|$. Then there exists a continuous function $x^*(t) : [c,d] \mapsto B_{X^*}$ such that $x^*(c) = x_c^*, x^*(d) = x_d^*$, and $\langle x^*(t), x(t)\rangle \geq \frac{1}{2}\|x(t)\|$ for every $t \in [c,d]$.*

Proof. Since $x(t)$ is continuous and not equal to zero there is a constant $C > 0$ such that $\|x(t)\| \geq C$ on $[c,d]$. Since $x(t)$ is absolute continuous we can find δ such that whenever $t, t' \in [c,d]$ and $|t - t'| < \delta$ we have $\|x(t) - x(t')\| < \frac{C}{7}$. We fix points $t_0 = c$, $t_1 = c + \delta$, $t_2 = c + 2\delta, ..., t_n = d$. We choose $x_0^* = x_c^*$ and $x_n^* = x_d^*$.

For each $1 \le i \le n-1$ take a real valued functional of norm one $x_i^* \in X^*$ such that $\langle x_i^*, x(t_i)\rangle \ge \frac{3}{4}\|x(t_i)\|$. Define $x^*(t) = sx_i^* + (1-s)x_{i+1}^*$ where $s \in [0,1]$ and $t = st_i + (1-s)t_{i+1}$, $0 \le i \le n$. Then $x^*(t)$ is continuous, $\|x^*(t)\| \le 1$. Take any $t \in [c,d]$ and find $0 \le i \le n$ and $s \in [0,1]$ so that $t = st_i + (1-s)t_{i+1}$. We have the following estimate

$$\begin{aligned}
\langle x^*(t), x(t)\rangle &\ge \langle sx_i^* + (1-s)x_{i+1}^*, sx(t_i) + (1-s)x(t_{i+1})\rangle \\
&\qquad - s\|x(t) - x(t_i)\| - (1-s)\|x(t) - x(t_{i+1})\| \\
&\ge \frac{3}{4}(s\|x(t_i)\| + (1-s)\|x(t_{i+1})\|) - \frac{C}{7} \\
&\ge \frac{3}{4}\|x(t)\| - \frac{3}{4}\frac{C}{7} - \frac{C}{7} \\
&\ge \frac{3}{4}\|x(t)\| - \frac{C}{4} \\
&\ge \frac{3}{4}\|x(t)\| - \frac{1}{4}\|x(t)\| \\
&= \frac{1}{2}\|x(t)\|.
\end{aligned}$$

□

Under mild assumptions on asymptotic behavior of $f, g \in H_0^\infty(\Sigma_\phi)$ with respect to the Boyd indices of E , we obtain that the norms on $X_E(f)$ and $X_E(g)$ are equivalent.

Theorem 4.8.6. *Assume that $f, g \in H_0^\infty(\Sigma_\phi)$ and satisfy the estimate*

$$|f(z)| \le C\frac{|z|^a}{(1+|z|)^{a+b}}, \quad |g(z)| \le C'\frac{|z|^{a'}}{(1+|z|)^{a'+b'}}$$

where $a, a' > max(0, -\alpha_E)$ and $b, b' > max(0, \beta_E)$. Then the norms $\|.\|_{X_E(f)}$ and $\|.\|_{X_E(g)}$ are equivalent.

Proof. Pick a_0, b_0 so that $max(-\alpha_E, 0) < a_0 < min(a, a')$ and $max(\beta_E, 0) < b_0 < min(b, b')$. It is enough to show that the norms $\|.\|_{X_E(f)}$ and $\|.\|_{X_E(\varphi_{a_0,b_0})}$ are equivalent. First, note that we can write $f(z) = \varphi_{a_0,b_0} h(z)$ where $h(z) = f(z)/\varphi_{a_0,b_0}(z)$. Then we have $|h(z)| = |f(z)z^{-a_0}(1+z)^{-a_0-b_0}| \leq C|z|^{a-a_0}(1+|z|^2)^{-(a+b-a_0-b_0)/2}$, and it follows that $h(z) \in H_0^\infty(\Sigma_\phi)$. For $x \in X$ we have

$$\begin{aligned}
\|x\|_{X_E(f)} &= \| \, \|f(tA)x\|_X \|_E \\
&\leq \| \, \|h(tA)\|_X \|\varphi_{a_0,b_0}(tA)x\|_X \|_E \\
&\leq \sup_{t>0} \|h(tA)\|_X \cdot \|x\|_{X_E(\varphi_{a_0,b_0})}
\end{aligned}$$

This shows the upper estimate of the desired equivalence.

For the lower estimate, fix a nonzero element $x \in X$. Since we have $\varphi_{a_0,b_0}(tA) = (-t)^{-b} A^a R(-\frac{1}{t}, A)^{a+b}$ and the operators A^a and $R(-\frac{1}{t}, A)$ are injective we obtain that $\varphi_{a_0,b_0}(tA)$ is injective as well. Therefore, $\varphi_{a_0,b_0}(tA)x \neq 0$ for all $t \in (0, \infty)$. From lemma 4.8.5 we are able to construct a continuous function $F^* : (0, \infty) \longrightarrow X^*$ such that $\|F^*(t)\| \leq 1$ for all t and

$$\langle \varphi_{a,b}(tA)x, F^*(t) \rangle \geq \frac{1}{2} \|\varphi_{a,b}(tA)x\|_X.$$

To see this, take dyadic intervals $[2^n, 2^{n+1}]$ and define F^* inductively with connecting endpoints.

Pick any $v \in E'$ supported in some compact subset of $(0, \infty)$ with $v \geq 0$ and $\|v\|_{E'} \leq 1$. Since $\|\varphi_{a_0,b_0}(\tau A)^* F^*(\tau) \frac{1}{\tau}\|$ is a continuous function on $(0, \infty)$ its restriction to the compact support of v belongs to E. Therefore we can compute the Bochner integral

$$x^* = \int_0^\infty v(\tau)\varphi_{a_0,b_0}(\tau A)^* F^*(\tau)\,\frac{d\tau}{\tau}.$$

Pick any $N > \max(a_0, b_0)$. Let us estimate $\|\varphi_{N,N}(tA)^* x^*\|$ using lemma 4.4

$$\begin{aligned}
\|\varphi_{N,N}(tA)^* x^*\| &= \|\int_0^\infty v(\tau)\varphi_{N,N}(tA)^*\varphi_{a_0,b_0}(\tau A)^* F^*(\tau)\,\frac{d\tau}{\tau}\| \\
&\le \int_0^\infty v(\tau)\|\varphi_{N,N}(tA)\varphi_{a_0,b_0}(\tau A)\|\,\frac{d\tau}{\tau} \\
&\le \int_0^\infty v(t\tau)\|\varphi_{N,N}(tA)\varphi_{a_0,b_0}(t\tau A)\|\frac{d\tau}{\tau} \\
&\le C_1\int_0^\infty v(t\tau)\frac{\tau^{a_0}}{(1+\tau)^{a_0+b_0}}(1+|\log\tau|)\frac{d\tau}{\tau}.
\end{aligned}$$

Choose $\alpha_E > a_1 > -a_0$ and $\beta_E < b_1 < b_0$. Using lemma (4.7.1) we obtain an estimate:

$$\begin{aligned}
\|\|\varphi_{N,N}(tA)^* x^*\|\|_{E'} &\le C_1\int_0^\infty \|v(t\tau)\|_{E'}\frac{\tau^{a_0}}{(1+\tau)^{a_0+b_0}}(1+|\log\tau|)\frac{d\tau}{\tau} \\
&\le C_1\int_0^\infty \|D_{\frac{1}{\tau}}\|_{E'}\|v\|_{E'}\frac{\tau^{a_0}}{(1+\tau)^{a_0+b_0}}(1+|\log\tau|)\frac{d\tau}{\tau} \\
&\le C_2\int_0^\infty \frac{\tau^{a_0+b_1}}{(1+\tau)^{a_0+a_1+b_0-b_1}}(1+|\log\tau|)\frac{d\tau}{\tau} \le C_3.
\end{aligned}$$

Let us now define $\tilde{f}(z) = \overline{f(\bar{z})}$. Then $\tilde{f}$ is analytic in the sector Σ and hence the following integral is independent of the path, thus we integrate along the positive real line.

$$\int_0^\infty f(tz)\tilde{f}(tz)\varphi_{N,N}(tz)\frac{dt}{t} = \int_0^\infty |f(t)|^2\varphi_{N,N}(t)\frac{dt}{t} = c > 0$$

for every z in a sector Σ_ϕ where $\phi > \omega$. It follows that if $y \in X$ and $y^* \in X^*$ then

$$|\langle y, y^*\rangle| \le c^{-1}\int_0^\infty |\langle f(tA)y, \tilde{f}(tA)^*\varphi_{N,N}(tA)^* y^*\rangle|\frac{dt}{t}.$$

Putting these remarks together we obtain:

$$\begin{aligned}\int_0^\infty v(t)\|\varphi(tA)x\|\frac{dt}{t} &\leq 2\int_0^\infty v(t)\langle x, F^*(t)\rangle\frac{dt}{t}\\ &\leq 2\langle x, x^*\rangle\\ &\leq 2c^{-1}\int_0^\infty |\langle f(tA)x, \tilde{f}(tA)^*\varphi_{N,N}(tA)^*x^*\rangle|\frac{dt}{t}\\ &\leq C_5\|x\|_{X_E(f)}\|\|(\tilde{f}(tA)\varphi_{N,N}(tA))^*x^*\|_{E'}\\ &\leq C_6\|x\|_{X_E(f)}\|(\varphi_{N,N}(tA))^*x^*\|_{E'}\\ &\leq C_7\|x\|_{X_E(f)}\end{aligned}$$

Taking the supremum over all choices of v gives

$$\|x\|_{X_E(\varphi_{a,b})} \leq C_7\|x\|_{X_E(f)}.$$

This completes the proof. $\square$

Now we can formulate the cental result of this section. Under mild assumptions on the dilation indices and growth bounds of $f \in H_0^\infty(\Sigma_\phi)$, a sectorial operator A always has absolute H^∞-calculus on $X_E(f)$.

Theorem 4.8.7. *Suppose that A is a sectorial operator on X and that E is an admissible function space. Let $f \in H_0^\infty(\phi)$ such that $|f(z)| \leq C\frac{|z|^a}{(1+|z|)^{a+b}}$ where $a > \max(0, -\alpha_E)$ and $b > \max(0, \beta_E)$. Then A has absolute calculus on $X_E(f)$.*

Proof. It follows from Theorem 4.8.6 that we can assume $f = \varphi_{a,b}$. Denote $d = \frac{1}{2(1+C_\phi)} + 1$, where C_ϕ is the constant of sectoriality.

We start with observation that for every $s > 0$ and $t \in (s, ds)$ the function $\varphi(sz)\varphi^{-1}(tz)$ corresponds to a bounded operator $\varphi(sA)\varphi^{-1}(tA)$. Moreover, the family of operators $\{\varphi(sA)\varphi^{-1}(tA) : t \in (s, ds)\}$ is uniformly bounded. To see this note that $|\varphi_{1,0}(sz)| = |\frac{sz}{1+sz}| < 1$ and write $\varphi(sz)\varphi^{-1}(tz)$ as a power series

$$\begin{aligned}\varphi(sz)\varphi^{-1}(tz) &= \left(\frac{s}{t}\right)^a \left(1 + \frac{t-s}{s}\frac{sz}{1+sz}\right)^{a+b} \\ &= \left(\frac{s}{t}\right)^a \sum_{k=0}^{\infty} \binom{a+b}{k} \left(\frac{t-s}{s}\varphi_{1,0}(sz)\right)^k .\end{aligned}$$

The operator $\varphi_{1,0}(sA)$ is bounded and we have the estimate

$$\|\frac{t-s}{s}\varphi_{1,0}(sA)\| \leq (d-1)\|I + \frac{1}{s}R(-\frac{1}{s}, A)\| \leq (d-1)(1 + C_\phi) \leq \frac{1}{2}.$$

Thus the operator series $\sum_{k=0}^{\infty} \binom{a+b}{k} \left(\frac{t-s}{s}\varphi_{1,0}(sA)\right)^k$ converges and we have the norm estimate

$$\|\varphi(sA)\varphi^{-1}(tA)\| \leq \left(\frac{1}{d}\right)^a \sum_{k=0}^{\infty} \binom{a+b}{k} \left(\frac{1}{2}\right)^k = C_1$$

Now for every $x \in X_E(f)$ and $t \in (s, ds)$ we obtain

$$\|\varphi_{a,b}(sA)x\|_X = \|\varphi_{a,b}(sA)\varphi_{a,b}^{-1}(tA)\varphi_{a,b}(tA)x\|_X \leq C_1\|\varphi_{a,b}(tA)x\|_X$$

Therefore for every $s > 0$

$$\|\chi_{(s,ds)}(t)\|\varphi_{a,b}(sA)x\|_X\|_E \leq C_1\|\|\varphi(tA)x\|_X\|_E \leq \|x\|_{X_{E(\varphi_{a,b})}} \tag{4.7}$$

We fix $s > 0$ and consider function $\varphi_{2\gamma,2\gamma}$ with $\gamma > max(a, b)$. Then

$$\begin{aligned}\|\varphi_{2\gamma,2\gamma}(sA)x\|_{X_{E(\varphi_{a,b})}} &= \|\ \|\varphi_{a,b}(tA)\varphi_{2\gamma,2\gamma}(sA)x\|_X\|_E \\ &= \|\ \|\varphi_{a,b}(tA)\varphi_{\gamma,\gamma}(sA)\varphi_{\gamma,\gamma}(sA)x\|_X\|_E \\ &\leq \|\ \|\varphi_{a,b}(tA)\varphi_{\gamma,\gamma}(sA)\|_X\|_E \cdot \|\varphi_{\gamma,\gamma}(sA)x\|_X\end{aligned}$$

Using Lemma 4.6.4 we obtain

$$\begin{aligned}\| \|\varphi_{a,b}(tA)\varphi_{\gamma,\gamma}(sA)\|_X\|_E\| \|\varphi_{a,b}(s\frac{t}{s}A)\varphi_{\gamma,\gamma}(sA)\|_X\|_E \\ \leq \|\frac{(t/s)^\gamma}{(1+t/s)^{2\gamma}}(1+|\log(t/s)|)\|_E \\ = \|\sum_{n=-\infty}^{\infty}\chi_{[d^ns,d^{n+1}s]}(t)\frac{(t/s)^\gamma}{(1+t/s)^{2\gamma}}(1+|\log(t/s)|)\|_E\end{aligned}$$

Since the function $\varphi(t/s) = \frac{(t/s)^\gamma}{(1+t/s)^{2\gamma}}$ is increasing for $0 < t < s$ and decreasing for $t > s$ we can replace it on each interval by its value at end-points. Then

$$\frac{(t/s)^\gamma}{(1+t/s)^{2\gamma}}(1+|\log(t/s)|) \leq \frac{d^{(n+1)\gamma}}{(1+d^{n+1})^{2\gamma}}(1+(n+1)|\log d|)$$

for $t > s$ and

$$\frac{(t/s)^\gamma}{(1+t/s)^{2\gamma}}(1+|\log(t/s)|) \leq \frac{d^{n\gamma}}{(1+d^{n})^{2\gamma}}(1+n|\log d|)$$

for $t < s$.

Hence we obtain

$$\begin{aligned}\|\sum_{n=-\infty}^{\infty}\chi_{[d^ns,d^{n+1}s]}(t)\frac{(t/s)^\gamma}{(1+t/s)^{2\gamma}}(1+|\log(t/s)|)\|_E \\ \leq C\cdot\|\sum_{n=-\infty}^{\infty}\chi_{[d^ns,d^{n+1}s]}\cdot\frac{d^{n\gamma}}{(1+d^n)^{2\gamma}}(1+n|\log d|)\|\end{aligned}$$

Now apply 4.7.1 and see that

$$\|\chi_{[d^ns,d^{n+1}s]}\| = \|D_{d^{-n}}\chi_{[s,ds]}\| \leq C(1+d^n)^{b-a}d^{-nb}$$

Therefore,

$$\begin{aligned}
&\|\varphi_{2\gamma,2\gamma}(sA)x\|_{X_{E(\varphi_{a,b})}} \\
&\qquad \leq C\|\chi_{[s,ds]}\| \sum_{n=-\infty}^{\infty} \frac{d^{n(\gamma-b)}}{(1+d^n)^{2\gamma-b+a}}(1+n|\log d|) \\
&\qquad \leq C\|\chi_{[s,ds]}\|(\sum_{n<0} d^{n(\gamma-b)}(1+n|\log d|) + \sum_{n\geq 0} d^{-n(\gamma+a)}(1+n|\log d|)) \\
&\qquad \leq C\|\chi_{[s,ds]}\|
\end{aligned}$$

We deduce

$$\|\varphi_{2\gamma,2\gamma}(sA)x\|_{X_{E(\varphi_{a,b})}} \leq C_2\|\chi_{[s,ds]}\|_E\|\varphi_{\gamma,\gamma}(sA)x\|_X \tag{4.8}$$

Suppose there are elements $x, y \in X_{E(\varphi_{a,b})}$ that satisfy

$$\|\varphi_{\delta,\delta}(sA)\varphi_{\gamma,\gamma}(sA)x\|_{X_{E(\varphi_{a,b})}} \leq \|\varphi_{\gamma,\gamma}(sA)y\|_{X_{E(\varphi_{a,b})}} \text{ whenever } 0 < s < \infty$$

Combining (4.7) and (4.8) we obtain

$$\begin{aligned}
C_1\|\chi_{(s,ds)}\|_E\|\varphi_{a+\delta+2\gamma,a+\delta+2\gamma}(sA)x\|_X &\leq \|\varphi_{\delta,\delta}(sA)\varphi_{2\gamma,2\gamma}(sA)x\|_{X_{E(\varphi_{a,b})}} \\
&\leq \|\varphi_{2\gamma,2\gamma}(sA)y\|_{X_{E(\varphi_{a,b})}} \\
&\leq C_2\|\chi_{(s,ds)}\|_E\|\varphi_{\gamma,\gamma}(sA)y\|_X
\end{aligned}$$

Thus for every $0 < s < \infty$ we have $C_1\|\varphi_{a+\delta+2\gamma,a+\delta+2\gamma}(sA)x\|_X \leq C_2\|\varphi_{\gamma,\gamma}(sA)y\|_X$. Since E is a Lattice we get

$$C_1\|x\|_{X_{E(\varphi_{a+\delta+2\gamma,a+\delta+2\gamma})}} \leq C_2\|y\|_{X_{E(\varphi_{\gamma,\gamma})}}.$$

The result now follows from Theorem 4.8.6. □

4.9 Absolute calculus and interpolation spaces

We are going to give a characterization of those spaces which allow operators to have absolute H^∞-calculus. These spaces are certain real interpolation spaces, hence we recall the K-method.

A Banach space X is an *interpolation space* between Banach spaces X_0 and X_1 if $X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1$ and every linear map $T : X_0 + X_1 \longrightarrow X_0 + X_1$ which is a bounded operator on X_0 and X_1 is also a bounded operator on X. An interpolation space X is called *exact* if we have the following norm estimate on T

$$\|T\|_X \leq \max\{\|T\|_{X_0}, \|T\|_{X_1}\}.$$

Given an interpolation couple (X_0, X_1) the standard construction of a real interpolation space $(X_0, X_1)_{\theta,q}$ uses the K-functional

$$K(t, x) = \inf\{\|x_0\|_{X_0} + t\|x_1\|_{X_1} : x = x_0 + x_1\}$$

and the weighted L_q-norm. Replacing the L_q-space by a Banach lattice E on $(\mathbb{R}_+, dt/t)$ with the assumption $\min(1, t) \in E$ generalizes this method [29, 14]. We define an interpolation space $(X_0, X_1)_E$ as the space of all $x \in X_0 + X_1$ such that $K(t, x) \in E$ and we can put $\|x\|_{(X_0,X_1)_E} = \|K(t, x)\|_E$.

Let us denote by

$$L_\infty = \{f \in L_0(\mathbb{R}_+, dt/t) : \text{ess sup}|f(t)| < \infty\}, \quad \|f\|_{L_\infty} = \text{ess sup}|f(t)|$$

and

$$L^1_\infty = \{f \in L_0(\mathbb{R}_+, dt/t) : \text{ess sup}|t^{-1}f(t)| < \infty\}, \quad \|f\|_{L^1_\infty} = \text{ess sup}|t^{-1}f(t)|.$$

Brudnyj and Krugljak showed [14, section 3.3] that it is sufficient to limit ourselves to exact interpolation spaces between L_∞ end L^1_∞.

Theorem 4.9.1 (Brudnyi-Krugljak). *Suppose* (X_0, X_1) *is an interpolation couple and E is a Banach lattice on* $(\mathbb{R}_+, dt/t)$ *with* $min(1, t) \in E$*. Then there exist* *an exact interpolation space F of the couple (L_∞, L^1_∞) such that spaces $(X_0, X_1)_E$ and $(X_0, X_1)_F$ are isomorphic.*

For further considerations it is important to know the Boyd indices for such interpolation spaces. Notice that $\|D_s\|_{L_\infty} = 1$ and $\|D_s\|_{L^1_\infty} = s^{-1}$ since $\|D_s f\|_{L^1_\infty} = \text{ess sup}_t |t^{-1} f(t/s)| = s^{-1} \text{ess sup}_{t/s} |(t/s)^{-1} f(t/s)| = s^{-1} \|f\|_{L^1_\infty}$. Suppose F is an exact interpolation space between L_∞ and L^1_∞ then $\|D_s\|_F \leq \max\{1, s^{-1}\}$ and we can estimate

$$\alpha_F = \liminf_{s \to 0} \frac{\log \|D_s\|_F}{\log s} \geq \liminf_{s \to 0} \frac{\log s^{-1}}{\log s} = -1$$
$$\beta_F = \limsup_{s \to \infty} \frac{\log \|D_s\|_F}{\log s} \leq \limsup_{s \to \infty} \frac{\log 1}{\log s} = 0$$

Therefore $-1 \leq \alpha_F \leq \beta_F \leq 0$.

Let A be a sectorial operator on a Banach space X. For any number $a \in \mathbb{R}$ one can define the fractional power A^a as a closed injective operator with dense domain and range (see [35, section 15] for details). We denote by $\tilde{\mathcal{D}}(A^a)$ the completion of $\mathcal{D}(A^a)$ under the norm $\|x\|_{\tilde{\mathcal{D}}(A^a)} = \|A^a x\|$.

Denote by $\mathcal{F}$ the set of all exact interpolation spaces F of (L_∞, L^1_∞) with $-1 < \alpha_F < \beta_F < 0$ such that F has the Fatou property and $L_\infty \cap L^1_\infty$ is dense in F.

Theorem 4.9.2. *Let A be a sectorial operator on X. Then A has absolute calculus on any interpolation space $(\tilde{\mathcal{D}}(A^{-b}), \tilde{\mathcal{D}}(A^a))_F$ where $F \in \mathcal{F}$ and $a, b > 0$.*

Proof. Suppose that A is a sectorial operator on X. Pick any $F \in \mathcal{F}$ and $a, b > 0$. We are going to show that A has absolute H^∞-calculus on $(\tilde{\mathcal{D}}(A^{-b}), \tilde{\mathcal{D}}(A^a))_F$.

First we check that $(\tilde{\mathcal{D}}(A^{-b}), \tilde{\mathcal{D}}(A^{a}))$ is a linearizable interpolation couple. This can be seen by defining the parameterized families of operators

$$V_0(t) : X \longrightarrow \tilde{\mathcal{D}}(A^{-b}), \quad V_0(t) = tA^{a+b}(I + tA^{a+b})^{-1}$$

and

$$V_1(t) : X \longrightarrow \tilde{\mathcal{D}}(A^{a}), \quad V_1(t) = (I + tA^{a+b})^{-1}.$$

Then $V_0(t) + V_1(t) = I$. Since A is sectorial we have for $x \in \mathcal{D}(A^{-b})$ that

$$\|V_0(t)x\|_{\tilde{\mathcal{D}}(A^{-b})}\|tA^{a+b}(I + tA^{a+b})^{-1}A^{-b}x\|_X \leq C\|A^{-b}x\|_X C\|x\|_{\tilde{\mathcal{D}}(A^{-b})}$$

and also

$$\|V_1(t)x\|_{\tilde{\mathcal{D}}(A^{a})} = \|A^{a}(I + tA^{a+b})^{-1}x\|_X \leq Ct^{-1}\|A^{-b}x\|_X = Ct^{-1}\|x\|_{\tilde{\mathcal{D}}(A^{-b})}$$

Similarly, for $x \in \mathcal{D}(A^{a})$ we get

$$\|V_0(t)x\|_{\tilde{\mathcal{D}}(A^{-b})} = \|tA^{a}(I + tA^{a+b})^{-1}x\|_X \leq Ct\|A^{a}x\|_X Ct\|x\|_{\tilde{\mathcal{D}}(A^{a})}$$

and

$$\|V_1(t)x\|_{\tilde{\mathcal{D}}(A^{a})} = \|A^{a}(I + tA^{a+b})^{-1}x\|_X \leq C\|A^{a}x\|_X = C\|x\|_{\tilde{\mathcal{D}}(A^{a})}$$

Therefore $(\tilde{\mathcal{D}}(A^{-b}), \tilde{\mathcal{D}}(A^{a}))$ is linearizable and hence by [52] we have the following estimate on the K-functional $K(t,x) \leq \|V_0(t)x\|_{\tilde{\mathcal{D}}(A^{-b})} + t\|V_1(t)x\|_{\tilde{\mathcal{D}}(A^{a})} \leq 2CK(x,t)$. Thus by the choice of V_0 and V_1 we have

$$\frac{1}{2}K(t,x) \leq \|tA^{a}(I + tA^{a+b})^{-1}x\|_X \leq CK(x,t)$$

We define a new Banach lattice

$$E = \{f \in L_0(0,\infty) : t^{\frac{b}{a+b}} f(t^{\frac{1}{a+b}}) \in F\}, \quad \|f(t)\|_E = \|t^{\frac{b}{a+b}} f(t^{\frac{1}{a+b}})\|_F.$$

It easy to check that E is a Köthe function space with Fatou property and we have $\| \|tA^a(I+tA^{a+b})^{-1}x\|_X\|_F = \| \|\varphi_{a,b}(tA)x\|_X\|_E$.

We can express the norm of the dilation operator D_s on E in terms of the norm of D_s on F.

$$\|D_s f(t)\|_E = \|f(t/s)\|_E = \|t^{\frac{b}{a+b}} f(t^{\frac{1}{a+b}}/s)\|_F = s^b \|D_{s^{a+b}}(t^{\frac{b}{a+b}} f(t^{\frac{1}{a+b}}))\|_F$$

Therefore $\|D_s\|_E = s^b\|D_{s^{a+b}}\|_F$. For the Boyd indices we obtain

$$\beta_E = \limsup_{s\to\infty} \frac{\log \|D_s\|_E}{\log s} = \limsup_{s\to\infty} \frac{\log(s^b\|D_{s^{a+b}}\|_F)}{\log s} = b + (a+b)\beta_F$$

and

$$\alpha_E = \liminf_{s\to 0} \frac{\log \|D_s\|_E}{\log s} = \liminf_{s\to 0} \frac{\log(s^b\|D_{s^{a+b}}\|_F)}{\log s} = b + (a+b)\alpha_F.$$

Therefore, E is an admissible function space. We will show that space $X_E(\varphi_{a,b})$ actually coincides with $(\tilde{\mathcal{D}}(A^{-b}), \tilde{\mathcal{D}}(A^a))_F$. In view of Definition 4.8.2 we need to verify that every $x \in (\tilde{\mathcal{D}}(A^{-b}), \tilde{\mathcal{D}}(A^a))_F$ satisfies

$$\lim_{s\to 0} \|s(s+A)^{-1}x\|_{X_E(\varphi_{a,b})} = \lim_{s\to\infty} \|A(s+A)^{-1}x\|_{X_E(\varphi_{a,b})} = 0$$

To see this, it suffices to consider a dense subset of $(\tilde{\mathcal{D}}(A^{-b}), \tilde{\mathcal{D}}(A^a))_F$. We can choose $\mathcal{D}(A^{-b}) \cap \mathcal{D}(A^a)$ since it is dense in $(\tilde{\mathcal{D}}(A^{-b}), \tilde{\mathcal{D}}(A^a))_F$ by [14, section 3.6].

Note that if $x \in \mathcal{D}(A^{-b}) \cap \mathcal{D}(A^a)$ then $s(s+A)^{-1}x$ and $A(s+A)^{-1}x$ are also in $\mathcal{D}(A^{-b}) \cap \mathcal{D}(A^a)$. This is due to the commutativity properties of extended functional calculus [35, section 15.B], e.g. $A^{-b}(s(s+A)^{-1}x) = s(s+A)^{-1}(A^{-b}x)$.

For $x \in \mathcal{D}(A^{-b}) \cap \mathcal{D}(A^a)$ we have

$$\|x\|_{(\tilde{\mathcal{D}}(A^{-b}),\tilde{\mathcal{D}}(A^a))_F} \leq \|x\|_{\mathcal{D}(A^{-b})\cap\mathcal{D}(A^a)} = \max\{\|A^{-b}x\|_X, \|A^a x\|_X\}$$

Therefore, it remains to show that

$$\lim_{s\to 0} \|A^{-b}s(s+A)^{-1}x\|_X = \lim_{s\to 0} \|A^a s(s+A)^{-1}x\|_X = 0$$

and

$$\lim_{s\to\infty} \|A^{-b}A(s+A)^{-1}x\|_X = \lim_{s\to\infty} \|A^a A(s+A)^{-1}x\|_X = 0.$$

To see that these limits hold, we recall that A is sectorial on X and apply Reference 4.8.4 to $A^{-b}x$ and $A^a x$.

Therefore, we have that the interpolation space $(\tilde{\mathcal{D}}(A^{-b}), \tilde{\mathcal{D}}(A^a))_F$ is isomorphic to $X_E(\varphi_{a,b})$. Then by Theorem 4.8.7, A has absolute calculus on the interpolation space provided that $a > -\alpha_E$ and $b > \beta_E$. This follows from $\alpha_F > -1$ and $\beta_F < 0$ together with the expression above relating α_E, β_E to α_F, β_F. Now, the proof is complete.

Remark 4.9.3. The condition that F is an exact interpolation space between L_∞ and L^1_∞ with dense intersection $L_\infty \cap L^1_\infty$ was needed to conclude that $\mathcal{D}(A^{-b}) \cap \mathcal{D}(A^a)$ is dense in $(\tilde{\mathcal{D}}(A^{-b}), \tilde{\mathcal{D}}(A^a))_F$. Therefore, we can reformulate the previous theorem in the following way:

Let A be a sectorial operator on X. Suppose F is an admissible function space with $-1 < \alpha_F < \beta_F < 0$. If $\mathcal{D}(A^{-b}) \cap \mathcal{D}(A^a)$ is dense in $(\tilde{\mathcal{D}}(A^{-b}), \tilde{\mathcal{D}}(A^a))_F$ then A has absolute calculus on $(\tilde{\mathcal{D}}(A^{-b}), \tilde{\mathcal{D}}(A^a))_F$ whenever $a, b > 0$.

Corollary 4.9.4. *Suppose A is a sectorial operator on X and $a, b > 0$. Then A has absolute calculus on real interpolation spaces $(\tilde{\mathcal{D}}(A^{-b}), \tilde{\mathcal{D}}(A^a))_{\theta,p}$ for $\theta \in (0,1)$ and $1 < p < \infty$*

Proof. It is obvious that $L_p(t^{-\theta p}dt/t)$ is an admissible function space and by Remark 4.7.2 $\alpha_{L_p} = \beta_{L_p} = -\theta$. The density of the intersection of the domains follows from the well known fact for real interpolation spaces (see e.g. [52, section 1.6]). □

Conversely, if A has absolute calculus on X then X is isomorphic to a real interpolation space of the form 4.9.2.

Theorem 4.9.5. *Suppose A has absolute calculus on X for function $g(z) = \varphi_{a,b}(z)$ in the definition 4.6.1. Then X is a real interpolation space between $\tilde{\mathcal{D}}(A^a)$ and $\tilde{\mathcal{D}}(A^{-b})$.*

Since A has absolute calculus, there are $\delta > 0, a > 0$, and $b > 0$ such that

$$\|\varphi_{\delta,\delta}(tA)\varphi_{a,b}(tA)x\| \leq \|\varphi_{a,b}(tA)y\| \qquad 0 < t < \infty \;\Rightarrow \|x\| \leq C\|y\|.$$

The following estimate of the K-functional for $(\tilde{\mathcal{D}}(A^{-b}), \tilde{\mathcal{D}}(A^a))$ was obtained in the proof of the previous theorem.

$$\frac{1}{2}K(t,x) \leq \|tA^a(I + tA^{a+b})^{-1}x\|_X \leq CK(x,t)$$

Now we verify that X is K-monotone, i.e. there exist $C > 0$ such that whenever $x, y \in X$ with $K(t,x) \leq K(t,y)$ for all $t > 0$, then $\|x\| \leq C\|y\|$. We are going to use the result of Brudnyi and Krugljak ([14, chapter 3], [29, chapter 26]) that says every K-monotone interpolation space can be obtained by K-method.

Suppose we have $K(t,x) \leq K(t,y)$ for $t > 0$. Then using uniform boundedness of $\varphi_{\delta,\delta}(tA)$ and lemma 4.8.1 together with the observations above we obtain

$$\begin{aligned}
\|\varphi_{\delta,\delta}(tA)\varphi_{a,b}(tA)x\| &\leq C\|\varphi_{a,b}(tA)x\| \leq C\|(tA)^a(I+(tA)^{a+b})^{-1}x\| \\
&\leq Ct^{-b}K(t^{a+b},x) \\
&\leq Ct^{-b}K(t^{a+b},y) \\
&\leq C\|(tA)^a(I+(tA)^{a+b})^{-1}y\| \\
&\leq C\|\varphi_{a,b}(tA)y\|.
\end{aligned}$$

It follows from the absolute calculus of A that $\|x\| \leq C\|y\|$. By the theorem of Brudnyi and Krugljak mentioned above we conclude that there exists a Banach lattice F so that

$$\|x\|_X \approx \|K(t,x)\|_F.$$

□

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VITA

Tamara Kucherenko was born in Kharkov, Ukraine, on January 30, 1979. After attending Kharkov State University Lyceeum she entered undergraduate studies at Kharkov National University in 1996. She completed her diploma in mathematics under the supervision of Dr. V. Kadets in June 2001. Subsequently she continued to pursue graduate studies at the University of Missouri-Columbia in Fall 2001. From October 2003 until February 2004, Tamara received a five months scholarship from the German Academic Exchange Service (DAAD) to conduct research with Pr. Dr. L. Weis at the University of Karlsruhe, Germany. Thereafter she spent two semesters at the University of South Carolina. With guidance of N. Kalton, Tamara plans to receive a doctoral degree in July 2005.