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J. Math. Anal. Appl. ●●● (●●●●) ●●●–●●●

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Real interpolation of domains of sectorial operators on L_p-spaces

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Received 5 October 2004

Submitted by J. Diestel

Abstract

Let A be a sectorial operator on a non-atomic L_p-space, $1 \leqslant p < \infty$, whose resolvent consists of integral operators, or more generally, has a diffuse representation. Then the fractional domain spaces $\mathcal{D}(A^\alpha)$ for $\alpha \in (0,1)$ do not coincide with the real interpolation spaces of $(L_q, D(A))$. As a consequence, we obtain that no such operator A has a bounded H^∞-calculus if $p = 1$.

Keywords: Sectorial operators; Real interpolation; Representation of regular operators; H^∞-calculus

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1. Introduction

It is not uncommon that properties of the Laplace operator extend to a sectorial operator A which satisfies a pointwise kernel bound of the kind

$$\left|(\lambda + A)^{-1} f(u)\right| \leqslant \int_{\Omega} k_\lambda(u,v)\left|f(v)\right| dv, \quad u \in \mathbb{R}^n, \tag{1.1}$$

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[1] The first author acknowledges support from DAAD scholarship 2003–2004.

[2] The second author acknowledges support from DFG (We 2847/1-1).

0022-247X/$ – see front matter © 2005 Published by Elsevier Inc.
doi:10.1016/j.jmaa.2005.02.009

for $f \in L_q$ and λ in a sector about $\mathbb{R}_+$. Here, k_λ is the kernel of $(\lambda - \Delta)^{-1}$ or a more general Poisson bound. In the case of $1 < q < \infty$, (1.1) implies that $(-A)$ has maximal L_p-regularity (see, e.g., [6], [10, Section 5]), or that A has a H^∞-functional calculus if A has one on L_2 [4], [10, Section 5]. In this paper we exhibit two more examples of such phenomena.

It is well known that Laplace operator on $L_1(\mathbb{R}^n)$ does not have a bounded H^∞-calculus. In Corollary 3.3 we show that if $p = 1$ then (1.1) implies that A does not have a bounded H^∞-functional calculus. This is still true if k_λ is the kernel of any positive integral operator on $L_1(\Omega)$ or if $(\lambda + A)^{-1}$ has a "diffuse representation" (see the definition below). If $(-A)$ generates a weakly compact semigroup this result is already contained in [5]. It seems remarkable that the very same estimate (1.1) that guarantees the boundedness of the H^∞-calculus in so many cases if $q \in (1, \infty)$, absolutely excludes it if $q = 1$.

It is also well known that for Δ on $L_q(\mathbb{R}^n)$, $1 < q < \infty$, $q \neq 2$ the fractional domains $D((1 - \Delta)^\alpha)$ are isomorphic to the Bessel potential spaces $W_q^{2\alpha}(\mathbb{R}^n)$. So they do not coincide with the real interpolation spaces $(L_q, \mathcal{D}(\Delta))_{\alpha,r}$ which are isomorphic to the Besov potential spaces $B_{q,r}^{2\alpha}(\mathbb{R}^n)$ (of course, they are the same for $q = 2$). In Theorem 3.1 we will show that (1.1) implies such a result for any sectorial operator A on L_q with $0 \in \rho(A)$ and $1 < q < \infty$, $q \neq 2$, i.e.

$$\mathcal{D}(A^\alpha) \neq \bigl(L_q, \mathcal{D}(A)\bigr)_{\alpha,r}, \quad 0 < \alpha < 1.$$

Again, it is enough that k_λ is the kernel of a positive integral operator on $L_q(\Omega)$, or that $(\lambda + 1)^{-1}$ has a diffuse representation. If we assume in addition that A has bounded imaginary powers it follows that the complex and real interpolation methods yield different results for the interpolation pair $(L_q, \mathcal{D}(A))$ (see Corollary 3.2).

Let us recall now some definitions. A closed operator A with domain $\mathcal{D}(A)$ is called *sectorial of type* ω if the spectrum $\sigma(A)$ is contained in a sector $\{z \in \mathbb{C}\colon |\arg(z)| < \omega\} \cup \{0\}$ and we have $\|\lambda R(\lambda, A)\| \leqslant C_\omega$ for $|\arg(\lambda)| > \omega$. We will write $\rho(A) = \mathbb{C} \setminus \sigma(A)$ for the resolvent set of A and $R(\lambda, A)$ for the resolvent at $\lambda \in \rho(A)$. Suppose that A is a sectorial operator of type ω and f is a holomorphic function on Σ_σ, where $\sigma > \omega$. Given that f satisfies the condition $\int_{\partial\Sigma_\delta} |f(\lambda)| \frac{1}{|\lambda|} |d\lambda| < \infty$, we can define

$$f(A) = \int\limits_{\partial\Sigma_\delta} f(\lambda) R(\lambda, A)\, d\lambda, \quad \omega < \delta < \sigma.$$

We say that A has *bounded* $H^\infty(\Sigma_\sigma)$*-functional calculus* if the map $f \mapsto f(A)$ can be extended to a bounded map from the space $H^\infty(\Sigma_\sigma)$ of bounded holomorphic functions on Σ_σ to $B(X)$ (see [9] for details).

For the definition of fractional powers in terms of the H^∞-calculus see, e.g., [10] and if $0 \in \rho(A)$ see also [12]. A sectorial operator A has bounded imaginary powers if A^{-it} for $t \in \mathbb{R}$ define bounded operators on X. Clearly, a bounded H^∞-calculus implies bounded imaginary powers.

For the most part we consider L_q-spaces on σ-finite non-atomic measure spaces $(\mathcal{K}, \mathfrak{B}, m)$ and (Ω, Σ, μ). We recall that a bounded operator T on L_q is positive if the image of every non-negative function is again a non-negative function. If an operator can be split into a difference of two positive operators then it is called *regular*. Regular operators between L_p spaces have a particularly useful representation (see [7,11,13]). Given a regular operator $T : L_p(K, m) \to L_q(\Omega, \mu)$ there is a family of regular Borel measures $(\nu_y(x))_{y\in\Omega}$ on K such that for every $f \in L_p(K, m)$ we have

$$Tf(y) = \int_K f(x)\,d\nu_y(x) \quad \mu\text{-a.e.}$$

Note that if all measures ν_y are absolutely continuous with respect to m then by the Radon–Nikodym theorem we obtain classical integral operators,

$$Tf(y) = \int_{\mathcal{K}} k(y, x)\,dm(x), \quad k(y, \cdot) = d\nu_y/dm.$$

In case that all measures ν_y are non-atomic we say that the operator has a *diffuse representation*.

While resolvents of second order elliptic operators are typically classical integral operators, the resolvents of first order differential operators have usually a diffuse representation. As an example, consider the operator $A : D(A) \supset L_1(\mathbb{R}^2) \to L_1(\mathbb{R}^2)$ given by

$$Af(x_1, x_2) = \frac{\partial}{\partial x_1} f(x_1, x_2).$$

Its resolvent

$$\bigl(R(A, \lambda)f\bigr)(x_1, x_2) = \int_0^\infty e^{-\lambda t} f(x_1 + t, x_2)\,dt$$

has representing measure

$$\mu^\lambda_{(x_1,x_2)} = \eta^\lambda_{x_1} \otimes \delta_{x_2},$$

where δ_{x_2} is the Dirac measure and $d\eta^\lambda_{x_1} = \chi_{[x_1,\infty)}(t)e^{-\lambda(t-x_1)}\,dt$. Therefore, $R(A, \lambda)$ is not an integral operator but has a diffuse representation. However, given a diffuse operator T, we can always pass to a sub-σ-algebra for which T is integral [13].

2. Preliminary results

The following lemma is a vector-valued version of a classical result about uniform integrability in L_1.

Lemma 2.1. *Let X be a Banach space and T be an isomorphic embedding from X into $L_p(X)$ $(1 \leqslant p < \infty)$. Assume that for some subspace $Y \subset X$ the set $\{\|Ty(t)\|^p_X\colon\ y \in Y,\ \|y\|_X = 1\}$ is not uniformly integrable as a subset of L_1. Then there exist a sequence (y_n) in Y isomorphic to a unit vector basis of l_p.*

Proof. Since $\{\|Ty(t)\|^p_X\colon\ y \in Y,\ \|y\|_X = 1\}$ is not uniformly integrable in L_1 we can find a sequence (y_n) in Y with $\|y_n\| \leqslant 1$ such that $\int \|Ty_n(t)\|^p_X\,dt = 1$ and $\|Ty_n(t)\|^p_X \to 0$ $(n \to \infty)$ almost everywhere. To see this, assume the contrary, i.e. every sequence from

$T(Y)$ converging to zero almost everywhere is converging to zero in $L_p(X)$-norm. Then for all $0 < q < p$ there exists $C > 0$ such that $\int \|Ty(t)\|^p\,dt \leqslant C \int \|Ty(t)\|^q\,dt$ for all $y \in Y$. Hence, we have

$$\begin{aligned}
&\lim_{M\to\infty} \sup_{\|y\|=1} \Biggl(\int\limits_{\|Ty(t)\|>M} \bigl\|Ty(t)\bigr\|^p\,dt \Biggr)^{1/p} \\
&\quad\leqslant C \lim_{M\to\infty} \sup_{\|y\|=1} \Biggl(\int\limits_{\|Ty(t)\|>M} \bigl\|Ty(t)\bigr\|^q\,dt \Biggr)^{1/q} \\
&\quad\leqslant C \lim_{M\to\infty} \sup_{\|y\|=1} \biggl(\int \bigl\|Ty(t)\bigr\|^p M^{q-p}\,dt \biggr)^{1/p} = 0.
\end{aligned}$$

This contradicts the fact that $\{\|Ty(t)\|_X^p\colon\ y \in Y,\ \|y\|_X = 1\}$ is not uniformly integrable in L_1.

For convenience define $f_n(t) = \|Ty_n(t)\|_X^p$. Then (f_n) are functions in L_1 of norm one. We will use a subsequence splitting lemma.

Lemma 2.2 [16]. *If (f_n) is a sequence in the unit ball of L_1 then there exist a subsequence (f_{n_k}) and disjoint sets (A_k) with their complements B_k such that $f_{n_k}|_{B_k}$ are uniformly integrable.*

Since the sequence $(f_{n_k}|_{B_k})$ is uniformly integrable and still goes to zero almost everywhere when k is approaching infinity we get that $f_{n_k}|_{B_k}$ goes to zero in L_1-norm. So $f_{n_k}|_{A_k}$ is bounded in norm from below. Now $Ty_{n_k} = Ty_{n_k}|_{B_k} + Ty_{n_k}|_{A_k}$, where $\|Ty_{n_k}|_{A_k}\|_{L_p(X)} = \|f_{n_k}|_{A_k}\|_{L_1}$ is bounded from below. Thus the sequence $(Ty_{n_k}|_{A_k})$ is isomorphic to the unit vector basis of l_p since it has disjoint support and bounded from below in $L_p(X)$. On the other hand

$$\|Ty_{n_k} - Ty_{n_k}|_{A_k}\|_{L_p(X)} = \|Ty_{n_k}|_{B_k}\|_{L_p(X)} = \|f_{n_k}|_{B_k}\|_{L_1} \to 0 \quad (k \to \infty).$$

It follows by perturbation of basis that some subsequence of (Ty_{n_k}) is equivalent to the unit vector basis of l_p. Denote this subsequence again by (Ty_{n_k}). Then (y_{n_k}) is also equivalent to the unit vector basis of l_p since T is an isomorphism. □

The next proposition is related to a result in [9]. The expression appearing in the statement will be applied to the setting of interpolation spaces between X and $D(A)$.

Proposition 2.3. *Suppose X is a Banach space and A is a sectorial operator on X. Assume there is a constant $C > 0$, $1 \leqslant p < \infty$ and $\alpha \in (0,1)$ such that for every $x \in X$,*

$$C^{-1}\|x\| \leqslant \Biggl(\int\limits_{-\infty}^{0} \bigl\| |t|^{\alpha-1/p} A^\alpha R(t,A)x \bigr\|^p\,dt \Biggr)^{1/p} \leqslant C\|x\|. \tag{2.1}$$

Then if Y is an infinite-dimensional closed subspace of $D(A)$ (with a graph norm) and Y does not contain a copy of l_p then A is bounded on Y.

Proof. We will consider an operator $T : X \mapsto L_p(\mathbb{R}\_, dt, X)$ given by

$$Tx(t) = |t|^{\alpha-1/p} A^\alpha R(t, A)x.$$

It follows from (2.1) that T is an isomorphic embedding. Since $\alpha < 1$ we can find a natural number m such that $\alpha \leqslant (m-1)/m$. Fix $s < 0$. Then $R(s, A)$ maps X isomorphically onto $D(A)$ (with a graph norm). Let $Y_0 = R(s, A)^{-1}Y$. Then Y_0 is an infinite-dimensional subspace of X that does not contain a copy of l_p. By Lemma 2.1 the set $\{\|Ty(t)\|_X^p\colon y \in Y,\ \|y\|_X = 1\}$ is uniformly integrable. The operator $A^\alpha R(s, A)$ has a lower bound on Y_0 since otherwise, there would exist a sequence y_n in Y_0 of elements of norm one such that $\|A^\alpha R(s, A)y_n\| \to 0$. However, the resolvent equation yields for any $t < 0$,

$$A^\alpha R(t, A)y_n = A^\alpha R(s, A)y_n + (s - t)R(t, A)\bigl(A^\alpha R(s, A)y_n\bigr).$$

Therefore $\|A^\alpha R(t, A)y_n\| \to 0$ pointwise. Now by uniform integrability and (2.1), we have $\|y_n\| \to 0$ which gives a contradiction.

The operator $A^\alpha R(s, A)$ is an isomorphism on Y_0. Thus the subspace

$$Y_1 = A^\alpha R(s, A)(Y_0)$$

does not contain a copy of l_p and by the same argument we get that $A^\alpha R(s, A)$ is bounded from below on Y_1. This gives us a lower bound for the operator $A^{2\alpha} R(s, A)^2$ on Y_0. Repeating the same procedure m times we get that the operator $A^{m\alpha} R(s, A)^m$ is bounded from below on X_0 by some constant $C > 0$. It follows from the boundedness of the operator $A^{m\alpha} R(s, A)^{m-1}$ $(\alpha \leqslant (m-1)/m)$ and the simple computation

$$C\|y_0\| \leqslant \bigl\|A^{m\alpha} R(s, A)^m y_0\bigr\| \leqslant \bigl\|A^{m\alpha} R(s, A)^{m-1}\bigr\| \bigl\|R(s, A)y_0\bigr\|, \quad y_0 \in Y_0,$$

that the resolvent $R(s, A)$ is bounded from below on Y_0.

Now we see that A is bounded on $Y = R(s, A)Y_0$. Take any y in Y and find y_0 in Y_0 such that $y = R(s, A)y_0$. Then

$$\begin{aligned}\|Ay\| &\leqslant \bigl\|AR(s, A)\bigr\| \|y_0\| \leqslant (1/C)\bigl\|AR(s, A)\bigr\| \bigl\|A^{m\alpha} R(s, A)^{m-1}\bigr\| \bigl\|R(s, A)y_0\bigr\| \\ &= C_1\|y\|. \quad \square\end{aligned}$$

Remark 2.4. The proposition cannot be applied for $p = 2$. In this case X is isomorphic to $L_2(\mathbb{R}\_, dt, X)$. Thus there is no subspace in X and hence in $\mathcal{D}(A)$ which does not contain a copy of l_2.

We assume that zero is contained in the resolvent set, then $(-\infty, 0] \subset \rho(A)$ and we have an estimate $\|R(t, A)\| \leqslant \frac{C}{1+|t|}$ for all $t \in (-\infty, 0]$. This allows us to apply a theorem from [12] which yields that an equivalent norm on the real interpolation space $(X, \mathcal{D}(A))_{\alpha,p}$ for $0 < \alpha < 1$ and $1 \leqslant p \leqslant \infty$ is given by

$$\|x\|_{(L_q, D(A))_{\alpha,p}} \approx \left(\int_0^\infty \bigl\|t^\alpha A(A+t)^{-1}x\bigr\|^p \frac{dt}{t}\right)^{1/p} \tag{2.2}$$

for $x \in (X, \mathcal{D}(A))_{\alpha,p}$.

In [9] it was shown that if A has an H^∞-calculus on L_1 then

$$\|x\|_{L_1} \approx \int_{-\infty}^{\infty} \|A^s R(t, A)x\| \frac{dt}{t}.$$

Formula (2.2) allows us to reformulate this statement as follows.

Proposition 2.5. *If A has a bounded H^∞-calculus on $L_1(\Omega, \mu)$ then $(L_1, \mathcal{D}(A))_{\alpha,1} = \mathcal{D}(A)^\alpha$ with equivalence of norms for $0 < \alpha < 1$.*

3. Main results

In general we have the following inclusions between the domain $\mathcal{D}(A^\alpha)$ of a fractional power of A and real interpolation spaces $(X, \mathcal{D}(A))_{\alpha,1}$ and $(X, \mathcal{D}(A))_{\alpha,\infty}$,

$$\big(X, \mathcal{D}(A)\big)_{\alpha,1} \subset \mathcal{D}(A^\alpha) \subset \big(X, \mathcal{D}(A)\big)_{\alpha,\infty}.$$

If a sectorial operator A has a bounded H^∞-calculus on $X = L_2(\mathcal{K}, \mathfrak{B}, m)$ then we have $\mathcal{D}(A^\alpha) = (X, \mathcal{D}(A))_{\alpha,2}$. This result can be found in [1]. As we will see now this statement is wrong for L_q with $q \neq 2$.

Theorem 3.1. *Let A be a sectorial operator on $L_q(\mathcal{K}, \mathfrak{B}, m)$ for a non-atomic measure space $(\mathcal{K}, \mathfrak{B}, m)$ and $1 < q < \infty$, $q \neq 2$. Assume that $0 \in \rho(A)$ and there exists $s < 0$ such that $R(s, A)$ is a regular operator with a diffuse representation. Then for any $\alpha \in (0, 1)$ and $1 \leqslant p \leqslant \infty$,*

$$D(A^\alpha) \neq \big(L_q, D(A)\big)_{\alpha,p}.$$

Proof. We will assume that $\mathcal{D}(A^\alpha) = (L_q, \mathcal{D}(A^\alpha))_{\alpha,p}$ and derive a contradiction.

It follows from [12] that there exists a constant $C > 0$ such that for any $y \in (L_q, D(A))_{\alpha,p}$ we have

$$C^{-1}\left(\int_0^\infty \|t^\alpha A(A+t)^{-1}y\|^p \frac{dt}{t}\right)^{1/p} \leqslant \|y\|_{(L_q,D(A))_{\alpha,p}}$$
$$\leqslant C\left(\int_0^\infty \|t^\alpha A(A+t)^{-1}y\|^p \frac{dt}{t}\right)^{1/p}.$$

Since $D(A^\alpha) = (L_q, D(A))_{\alpha,p}$, we obtain for any $y \in D(A^\alpha)$ that the quantities $\|A^\alpha y\|$, $\|y\|_{D(A^\alpha)}$, and $\|y\|_{(L_q,D(A))_{\alpha,p}}$ are equivalent.

Pick x from the range of A^α and take $y \in \mathcal{D}(A^\alpha)$ such that $x = A^\alpha y$. Then using $\int_0^\infty \|t^\alpha A(A+t)^{-1}A^{-\alpha}x\|^p \frac{dt}{t} = \int_{-\infty}^0 \||t|^{\alpha-1/p} A^{1-\alpha} R(t, A)x\|^p \, dt$ we obtain for some suitable constant $C > 0$ that

$$C^{-1}\left(\int_{-\infty}^{0}\left\||t|^{\alpha-1/p}A^{1-\alpha}R(t,A)x\right\|^{p}\,dt\right)^{1/p}\leqslant\|x\|_{L_q(\mathcal{K},\mathfrak{B},m)}$$

$$\leqslant C\left(\int_{-\infty}^{0}\left\||t|^{\alpha-1/p}A^{1-\alpha}R(t,A)x\right\|^{p}\,dt\right)^{1/p}.$$

The range of A^α is dense in $L_q(\mathcal{K},\mathfrak{B},m)$ and therefore condition (2.1) is fulfilled. We will use Proposition 2.3. Since the resolvent $R(s,A)$ is a regular operator with a diffuse representation there is a non-atomic sub σ-algebra $\mathfrak{B}_1$ of $\mathfrak{B}$ such that $R(s,A)|_{L_q(K,\mathfrak{B}_1,m)}$ is a compact operator [13]. Let Y_1 be a closed infinite-dimensional subspace of $L_q(\mathcal{K},\mathfrak{B}_1,m)$ which does not contain a copy of l_p, for instance, take the span of a sequence equivalent to the Rademacher functions. Consider $Y=R(s,A)Y_1$. Since $R(s,A)$ is an isomorphism from $L_q(\mathcal{K},\mathfrak{B},m)$ onto $D(A)$ (with the graph norm), Y is a closed infinite-dimensional subspace of $D(A)$ and does not contain l_p. By Proposition 2.3, A is bounded on Y and therefore $mI-A$ is also bounded on Y. We consider the bounded operator

$$J:\bigl(D(A),\|.\|_A\bigr)\to L_q(\mathcal{K},\mathfrak{B},m),\qquad J=R(s,A)(mI-A).$$

Then $J(Y)=Y$. On the other hand, $J|_Y=R(s,A)(s-A)=R(s,A)|_{Y_1}$ is a compact operator since $Y_1\subset L_q(\mathcal{K},\mathfrak{B}_1,m)$. This is impossible since J is onto Y and Y is infinite-dimensional. We hence obtain a contradiction. □

It is well known that if A has bounded imaginary powers on X then $\mathcal{D}(A^\alpha)$ coincides with the complex interpolation spaces $[X,\mathcal{D}(A)]^\alpha=\mathcal{D}(A)^\alpha$ (see, e.g., [10,12]). Hence our theorem implies

Corollary 3.2. *Assume in addition to the assumption of Theorem* 3.1 *that A has bounded imaginary powers. Then*

$$\bigl(L_p,\mathcal{D}(A)\bigr)_{\alpha,p}\neq\bigl[L_p,\mathcal{D}(A)\bigr]_\alpha$$

for all $1\leqslant p\leqslant\infty$ and $\alpha\in(0,1)$.

Our next results will show that no reasonable differential operator on $L_1(\Omega,\mu)$ can have a bounded H^∞-calculus.

Corollary 3.3. *Let A be a sectorial operator on $L_1(\Omega,\Sigma,\mu)$. Assume there is a point $\lambda\in\rho(A)$ such that the resolvent $R(\lambda,A)$ has a diffuse representation. Then A does not have a bounded H^∞-calculus.*

Proof. Combine Proposition 2.5 and Theorem 3.1 noticing that all operators on L_1 are regular. □

For a variant of our assumption recall the Sobolev spaces defined for $s\in\mathbb{R}$ and $1\leqslant p\leqslant\infty$ as

$$H_p^s=\bigl\{u\in\mathcal{S}':\ \bigl\|\mathcal{F}^{-1}\bigl\{\bigl(1+|\xi|^2\bigr)^{s/2}\mathcal{F}u(\xi)\bigr\}\bigr\|_{L_p}<\infty,$$

where $\mathcal{F}:\mathcal{S}' \to \mathcal{S}'$ denotes the Fourier transform for tempered distributions (see [2,12]).

Corollary 3.4. *Let $\Omega \subset \mathbb{R}^n$ with piecewise smooth boundary. Suppose that $A: L_1(\Omega) \supset \mathcal{D}(A) \to L_1(\Omega)$ is a sectorial operator such that $\mathcal{D}(A) \subset H_1^s(\Omega)$ for some $s > 0$. Then A does not have an H^∞-calculus.*

Proof. To apply Theorem 3.3 we need to show that $R(\lambda, A)$ has a diffuse representation for some $\lambda \in \rho(A)$. Pick any $\lambda \in \rho(A)$. Then by Sobolev's theorem we have a continuous inclusion $H_1^s(\Omega) \hookrightarrow L_p(\Omega)$ for some $p > 1$. Hence, for any bounded set $U \subset \Omega$ with piecewise smooth boundary we obtain that $\chi_U R(\lambda, A)$ factors through $L_p(U)$,

$$L_1(\Omega) \xrightarrow{\chi_U R(\lambda, A)} \mathcal{D}(A) \cap L_1(U) \hookrightarrow H_1^s(U) \hookrightarrow L_p(U) \hookrightarrow L_1(U).$$

Consequently, $\chi_U R(\lambda, A)$ is a weakly compact operator. Notice that $\mu(U)$ is finite. Therefore, $\chi_U R(\lambda, A)$ is an integral operator [3]. This argument works for all bounded $U \subset \Omega$ with piecewise smooth boundary and thus $R(\lambda, A)$ has a diffuse representation. According to Corollary 3.3, A does not have an H^∞-calculus. □

Uncited references

[8] [14] [15]

References

[1] P. Auscher, A. McIntosh, A. Nahmod, Holomorphic functional calculi of operators, quadratic estimates and interpolation, Indiana Univ. Math. J. 46 (1997) 375–403.
[2] J. Bergh, J. Löfström, Interpolation Spaces, Springer-Verlag, 1976.
[3] J. Diestel, J.J. Uhl, Vector measures, Math. Surveys 15 (1977) 59–96.
[4] X.T. Duong, D.W. Robertson, Semigroup kernels, Poisson bounds and holomorphic functional calculus, J. Funct. Anal. 142 (1996) 89–128.
[5] M. Hoffmann, N. Kalton, T. Kucherenko, R-bounded approximating sequences and applications to semigroups, J. Math. Anal. Appl. 294 (2004) 373–386.
[6] M. Hiber, J. Prüss, Heat kernels and maximal $L^p - L^q$ estimates for parabolic evolution equations, Comm. Partial Differential Equations 22 (1997) 1647–1669.
[7] N. Kalton, The endomorphisms of L_p $(0 \leqslant p \leqslant 1)$, Indiana Univ. Math. J. 27 (1978) 353–381.
[8] N. Kalton, Representations of operators between function spaces, Indiana Univ. Math. J. 33 (1984) 639–665.
[9] N. Kalton, L. Weis, The H^∞-calculus and sums of closed operators, Math. Ann. 321 (2001) 319–345.
[10] P. Kunstman, L. Weis, Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus, in: M. Iannelli, R. Nagel, S. Piazzere (Eds.), Functional Analytic Methods for Evolution Equations, Springer Lecture Notes, 2004, in press.
[11] A.R. Sourour, Pseudo-integral operators, Trans. Amer. Math. Soc. 253 (1979) 339–363.
[12] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, 1995, pp. 91–107.
[13] L. Weis, On the representation of order continuous operators by random measure, Trans. Amer. Math. Soc. 285 (1984) 535–563.
[14] L. Weis, A note on diffuse random measures, Z. Wahrsch. Verw. Gebiete 65 (1983) 239–244.
[15] L. Weis, A generalization of the Vidav–Jörgens perturbation theorem for semigroups and its application to transport theory, J. Math. Anal. Appl. 129 (1988) 6–23.
[16] W.B. Johnson, J. Lindenstrauss (Eds.), Handbook of the Geometry of Banach Spaces, vol. 1, North-Holland, Amsterdam, 2001, pp. 125–160.