

RADEMACHER BOUNDED FAMILIES OF OPERATORS ON L_1

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ABSTRACT. For an R-bounded families of operators on L_1 we associate a family of representing measures and show that they form a weakly compact set. We consider a sectorial operator A which generates an R-bounded semigroup on L_1 . We show that given $\epsilon > 0$ there is an invertible operator $U : L_1 \rightarrow L_1$ with $\|U - I\| < \epsilon$ such that for some positive Borel function w we have $U(\mathcal{D}(A)) \supset L_1(w)$.

1. INTRODUCTION

Recall that a closed operator A on a Banach space X is called sectorial if

- The domain $\mathcal{D}(A)$ and range $\mathcal{R}(A)$ are dense,
- A is one-to-one
- The spectrum $\sigma(A)$ is contained in a sector $\Sigma_\omega = \{\zeta : |\arg \zeta| \leq \omega\}$.
- The resolvent $R(\zeta, A)$ satisfies the estimate

$$\|\zeta R(\zeta, A)\| \leq C_\phi, \quad |\arg(\zeta)| \geq \phi$$

whenever $\omega < \phi < \pi$.

If $\omega < \frac{\pi}{2}$ then the operator $-A$ generates a bounded analytic semigroup, $T_u = e^{-uA}$, $u \in [0, 1]$. Conversely if $-A$ is the generator of a bounded analytic semigroup then A is sectorial with $\omega < \pi/2$, provided it is one-one. For further discussion on sectorial operators see [2].

A is called R -sectorial with angle of R -sectoriality $\omega_R = \omega_R(A)$ if for every $\phi > \omega_R$ the collection of operators $\{R(\zeta, A) : |\arg \zeta| \geq \phi\}$ is R -bounded. We recall here that a collection of operators \mathcal{T} on a Banach space X is called R -bounded if there is a constant C so that

$$(\mathbb{E} \|\sum_{j=1}^n \varepsilon_j T_j x_j\|^2)^{1/2} \leq C (\mathbb{E} \|\sum_{j=1}^n \varepsilon_j x_j\|^2)^{1/2} \quad x_1, \dots, x_n \in X, T_1, \dots, T_n \in \mathcal{T}.$$

Here $(\varepsilon_j)_{j=1}^\infty$ is a sequence of independent Rademacher functions. R -sectoriality has become a very important property for sectorial operators because of the theorem of Weis [9] which shows that in a Banach space with (UMD) this property is closely related to maximal regularity.

This note is concerned with the structure of R -sectorial operators on the Banach space $L_1 = L_1(K, \lambda)$ where K is a Polish space (i.e. a topological space which is homeomorphic to a separable complete metric space) and λ is a nonatomic σ -finite Borel measure. All such spaces are isometric to $L_1 = L_1[0, 1]$, and so we will assume that K is a compact metric space and λ is a probability measure.

If A is a sectorial operator on L_1 which has H^∞ -calculus (for some angle) then A is R -sectorial (for some angle). We refer to [8] for the definition and discussion of the H^∞ -calculus. In [8] it was shown that if A has an H^∞ -calculus then A is bounded on

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any reflexive subspace of $\mathcal{D}(A)$ (with the graph norm); this had the implication there are very few examples of sectorial operators with an H^∞ -calculus on L_1 and in particular essentially no reasonable differential operator can have this property. In [5] it was shown that there are no R-bounded strongly continuous semigroups on L_1 consisting of weakly compact operators; it also follows from the results of [5] that if A is an R-sectorial operator on L_1 then the resolvent $R(\zeta, A)$ can never be a weakly compact operator.

The simplest example of a sectorial operator on $L_1(K, \lambda)$ which has an H^∞ -calculus and hence is R-sectorial is the operator

$$Af(s) = b(s)f(s)$$

where $b > 0$ a.e. and with domain

$$\mathcal{D}(A) = \left\{ f : \int |f(s)|b(s)^{-1}d\lambda(s) < \infty \right\}.$$

Note here that the domain is very large indeed; in fact for any $\epsilon > 0$ we can find a Borel set B with $\lambda(B) > 1 - \epsilon$ and such that $L_1(B) \subset \mathcal{D}(A)$. Of course one can get further examples by considering $A' = UAU^{-1}$ for U any invertible operator with $\mathcal{D}(A') = U(\mathcal{D}(A))$.

In this note, we show that this example is typical. Precisely we show that if A is R-sectorial and $\epsilon > 0$ then there is an invertible operator $U : L_1 \rightarrow L_1$ with $\|U - I\| < \epsilon$ such that for some positive Borel function w we have $U(\mathcal{D}(A)) \supset L_1(w)$. This improves both the results of [5] and [8].

2. OPERATORS ON L_1

Let K be a compact metric space and suppose λ is a probability measure on K . We will utilize the so-called random measure representation of operators on L_1 , developed in [6], [4] and [10].

A random measure on K is a map $s \rightarrow \mu_s$ from K into $\mathcal{M}(K)$ which is Borel for the weak*-topology on $\mathcal{M}(K)$. If the random measure satisfies the condition

$$(2.1) \quad \int_K |\mu_s|(B)d\lambda(s) \leq C\lambda(B) \quad B \in \mathcal{B}$$

then it induces a bounded operator $T : L_1(\lambda) \rightarrow L_1(\lambda)$ given by the formula

$$(2.2) \quad Tf(s) = \int_K f(t)d\mu_s(t) \quad \lambda - \text{a.e.}$$

and then $\|T\| \leq C$.

Conversely every bounded linear operator $T : L_1(\lambda) \rightarrow L_1(\lambda)$ has an essentially unique random measure representation and $\|T\|$ is the least constant C so that (2.1) holds.

We may also associate to T a unique measure ρ_T on $K \times K$ given by

$$\rho_T(E) = \int_K \left(\int_K \chi_E(s, t)d\mu_s^T(t) \right) d\lambda(s) \quad E \in \mathcal{B}(K \times K).$$

Thus

$$\rho_T(A \times B) = \int_A T\chi_B d\lambda.$$

The map $T \rightarrow \rho_T$ maps $\mathcal{L}(L_1)$ onto an order-ideal in $\mathcal{M}(K \times K)$ consisting of all measures ρ such that

$$|\rho|(A \times B) \leq C\lambda(B), \quad A, B \in \mathcal{B}(K \times K).$$

The space $\mathcal{L}(L_1(K, \lambda))$ is a complex Banach lattice and it is easily checked that if $T \in \mathcal{L}(L_1)$ then $\mu_s^{|T|} = |\mu_s^T|$ (λ -a.e.) and that $\rho_{|T|} = |\rho_T|$. Since it is a Banach lattice we can define as usual an operator $(\sum_{j=1}^n |T_j|^2)^{\frac{1}{2}}$ for any $T_1, \dots, T_n \in X$.

The following result is implicitly contained in ideas of [6], and more explicitly in [7] :

Proposition 2.1. *Let $T_n : L_1 \rightarrow L_1$ be a uniformly bounded sequence of operators such that $\lim_{n \rightarrow \infty} \|\rho_{T_n}\| = 0$. Then given any $\epsilon > 0$ there is a Borel subset B of K with $\lambda(B) > 1 - \epsilon$ and $n \in \mathbb{N}$ so that we have*

$$\|T_n f\| \leq \epsilon \|f\| \quad f \in L_1(B).$$

Proof. Consider the measures on K given by

$$\nu_n(A) = |\rho_{T_n}|(A \times K).$$

Then ν_n is absolutely continuous with respect to λ . Let w_n be the Radon-Nikodym derivatives. Then

$$\int w_n d\lambda = |\rho_{T_n}|(K \times K) \rightarrow 0.$$

Therefore, $w_n \rightarrow 0$ in measure. Hence there exists $n \in \mathbb{N}$ and B with $\lambda(B) > 1 - \epsilon$ so that $|w_n| < \epsilon$ on B .

However

$$\|T_n\| = \|w_n\|_\infty.$$

This follows from $\|T_n\| = \sup_{\lambda(B) > 0} \frac{v_n(B)}{\lambda(B)}$, $v_n(B) = \int_B w_n d\lambda$, and $v_n(B) = |\rho_{T_n}|(B \times K) = \iint \chi_{B \times K} d|\mu_s^{T_n}|(t) = \int_K |\mu_s^{T_n}|(B) d\lambda$.

If $f \in L_1(B)$ we have

$$\|T_n f\| \leq \int_{K \times K} |f(s)| d|\rho_T|(s, t) \leq \int_B |f(s)| w_n(s) d\lambda(s) \leq \epsilon \|f\|.$$

□

If $T \in \mathcal{L}(L_1)$ then we can write μ_s as given in (2.2) in the form

$$\mu_s = a(s)\delta_s + \mu'_s \quad \lambda - \text{a.e.}$$

where $\mu'_s\{s\} = 0$ λ -a.e. and a is a bounded Borel function. (See for example [6]). Thus

$$Tf(s) = a(s)f(s) + \int_K f(t) d\mu'_s(t) \quad \lambda - \text{a.e.}$$

If we define the diagonal part of T by

$$\Pi(T)f = a(s)f(s)$$

then $\rho_{\Pi(T)}$ is the restriction of ρ_T to the diagonal subset $\Delta = \{(s, s) : s \in K\}$. Thus

$$\rho_{\Pi(T)}(B) = \rho_T(B \cap \Delta).$$

Theorem 2.2. *Let \mathcal{T} be a family of operators in $\mathcal{L}(L_1)$. Then the following are equivalent:*

- (i) \mathcal{T} is R -bounded.
- (ii) $\{(\sum_{j=1}^n a_j^2 |T_j|^2)^{1/2} : \sum_{j=1}^n |a_j|^2 \leq 1, T_k \in \mathcal{T}, n \in \mathbb{N}\}$ is uniformly bounded.

Proof. Assume \mathcal{T} is R -bounded so that

$$\mathbb{E} \left\| \sum_{j=1}^n \epsilon_j T_j x_j \right\| \leq M \mathbb{E} \left\| \sum_{j=1}^m \epsilon_j x_j \right\|$$

for any $T_1, \dots, T_n \in \mathcal{T}$ and $x_1, \dots, x_n \in X$. Suppose $T_1, \dots, T_n \in \mathcal{T}$ and $a_1, \dots, a_n \in \mathbb{C}$ are such that $\sum_{j=1}^n |a_j|^2 \leq 1$. Then, by Khintchine's inequality for lattices,

$$\left\| \left(\sum_{j=1}^n |a_j|^2 |T_j|^2 \right)^{\frac{1}{2}} \right\| \leq C \mathbb{E} \left\| \sum_{j=1}^n \epsilon_j a_j T_j \right\|$$

where C is an absolute constant.

Now choose any sequence of partitions $\mathcal{A}_m = (A_{mj})_{j=1}^{N_m}$ so that each \mathcal{A}_{m+1} refines \mathcal{A}_m and

$$\lim_{m \rightarrow \infty} \sup_{1 \leq j \leq N_m} \text{diam} A_{mj} = 0.$$

Then for any positive function $f \in L_1(K, \lambda)$ and any $T \in \mathcal{L}(L_1(K, \lambda))$ we have

$$|T|f = \lim_{m \rightarrow \infty} \sum_{j=1}^{N_m} |T(f\chi_{A_{mj}})| \quad \lambda - \text{a.e.}$$

Thus

$$\left| \sum_{k=1}^n \epsilon_k a_k T_k |f \right| = \lim_{m \rightarrow \infty} \sum_{j=1}^{N_m} \left| \sum_{k=1}^n \epsilon_k a_k T_k (f\chi_{A_{mj}}) \right| \quad \lambda - \text{a.e.}$$

Now by R-boundedness

$$\begin{aligned} \mathbb{E} \int_K \sum_{j=1}^{N_m} \left| \sum_{k=1}^n \epsilon_k a_k T_k (f\chi_{A_{mj}}) \right| d\lambda &= \sum_{j=1}^{N_m} \mathbb{E} \left\| \sum_{k=1}^n \epsilon_k a_k T_k (f\chi_{A_{mj}}) \right\| \\ &\leq M \sum_{j=1}^{N_m} \mathbb{E} \left\| \sum_{k=1}^n \epsilon_k a_k f \chi_{A_{mj}} \right\| \\ &\leq M \left(\sum_{k=1}^n |a_k|^2 \right)^{\frac{1}{2}} \sum_{j=1}^{N_m} \|f\chi_{A_{mj}}\| \\ &\leq M \|f\|. \end{aligned}$$

It follows from Fatou's Lemma that

$$\mathbb{E} \left\| \sum_{k=1}^n \epsilon_k a_k T_k \right\| \leq M$$

and hence

$$\left\| \left(\sum_{k=1}^n |a_k|^2 |T_k|^2 \right)^{\frac{1}{2}} \right\| \leq CM.$$

Now let us prove that (ii) \implies (i). First suppose $f \in L_1(K, \lambda)$ is positive and $T_1, \dots, T_n \in \mathcal{L}(L_1(K, \lambda))$. Then if $a_1, \dots, a_n \geq 0$ and $a_1^2 + \dots + a_n^2 = 1$ we have

$$\sum_{j=1}^n a_j |T_j| f \leq \left(\sum_{j=1}^n |T_j|^2 \right)^{\frac{1}{2}} f.$$

Now the least upper bound of the left hand-side over all choices of a_1, \dots, a_n is $(\sum_{j=1}^n (|T_j|f)^2)^{\frac{1}{2}}$ and so

$$\left(\sum_{j=1}^n (|T_j|f)^2 \right)^{\frac{1}{2}} \leq \left(\sum_{j=1}^n |T_j|^2 \right)^{\frac{1}{2}} f.$$

Let from our assumption (ii) there is a constant M so that

$$\left\| \left(\sum_{j=1}^n |a_j|^2 |T_j|^2 \right)^{\frac{1}{2}} \right\| \leq M \quad T_1, \dots, T_n \in \mathcal{T}, \quad |a_1|^2 + \dots + |a_n|^2 = 1.$$

Suppose $f \in L_1$ and $T_1, \dots, T_n \in \mathcal{T}$.

Then

$$\begin{aligned} \mathbb{E} \left\| \sum_{j=1}^n \epsilon_j a_j T_j f \right\| &\leq \left\| \left(\sum_{j=1}^n |a_j|^2 |T_j f|^2 \right)^{\frac{1}{2}} \right\| \\ &\leq \left\| \left(\sum_{j=1}^n |a_j|^2 (|T_j| |f|)^2 \right)^{\frac{1}{2}} \right\| \\ &\leq \left\| \left(\sum_{j=1}^n |a_j|^2 |T_j|^2 \right)^{\frac{1}{2}} |f| \right\| \\ &\leq M \|f\|. \end{aligned}$$

Now by Theorem 2.2 of [5] this means \mathcal{T} is R -bounded. \square

Proposition 2.3. *Suppose \mathcal{T} is an R -bounded family of operators on $L_1(K, \lambda)$. Then the family of measures $\{\rho_T : T \in \mathcal{T}\}$ is relatively weakly compact in $\mathcal{M}(K \times K)$.*

Proof. Note that if $T_1, \dots, T_n \in \mathcal{T}$ then

$$\left\| \max_{1 \leq k \leq n} |T_k| \right\| \leq \left\| \left(\sum_{k=1}^n |T_k|^2 \right)^{\frac{1}{2}} \right\| \leq Mn^{1/2}$$

where

$$M = \sup \left\{ \left\| \sum_{j=1}^n |a_j|^2 |T_j|^2 \right\|^{\frac{1}{2}} : T_1, \dots, T_n \in \mathcal{T}, \sum_{j=1}^n |a_j|^2 \leq 1 \right\}$$

which is finite by Theorem 2.2. The maximum here is computed in the lattice $\mathcal{L}(L_1)$.

Hence

$$\left\| \max_{1 \leq k \leq n} |\rho_{T_k}| \right\|_{\mathcal{M}(K \times K)} \leq Mn^{\frac{1}{2}}.$$

Assume the set $\{\rho_T : T \in \mathcal{T}\}$ is not relatively weakly compact. Then by Dieudonne-Grothendieck characterization [3] there is a $\delta > 0$, a sequence $(T_k)_{k=1}^n$ and a sequence of disjoint open sets U_k in $K \times K$ so that $\rho_{T_k}(U_k) \geq \delta$ for all k . Then

$$\left\| \max_{1 \leq k \leq n} |\rho_{T_k}| \right\|_{\mathcal{M}(K \times K)} \geq \sum_{k=1}^n \rho_{T_k}(U_k) \geq \delta n \quad n = 1, 2, \dots$$

which gives a contradiction. \square

3. APPLICATIONS TO SECTORIAL OPERATORS

Now we will give some applications of this theorem to sectorial operators.

Proposition 3.1. *If A is R -sectorial and $\omega_R(A) < \pi/2$ then $\{e^{-tA} : 0 < t < \infty\}$ is an R -bounded semigroup. Conversely if A is sectorial and $-A$ generates an R -bounded semigroup then A is R -sectorial with $\omega_R(A) \leq \pi/2$.*

If $-A$ is a sectorial operator which generates a semigroup $\{e^{-tA} : 0 < t < \infty\}$ with the property that $\{e^{-tA} : 0 < t \leq 1\}$ is R -bounded then for any $\phi > \pi/2$ there exists M so that the set $\{\zeta R(\zeta, A) : |\arg(\zeta + M)| \geq \phi\}$ is R -bounded.

Proof. These are simple deductions from the formulas:

$$\zeta R(\zeta, A) = \int_0^\infty \zeta e^{\zeta t} e^{-tA} dt$$

and

$$e^{-tA} - (1+tA)^{-1} = -\frac{1}{2\pi i} \int_{\Gamma_\nu} (e^{-t\zeta} - (1+t\zeta)^{-1}) R(\zeta, A) d\zeta$$

where Γ_ν is a contour of the form $\{|s|e^{i(\operatorname{sgn} s)\nu} : -\infty < s < \infty\}$ for any ν with $\nu > \omega(A)$. Indeed, assuming that $\{e^{-At} : 0 < t < \infty\}$ is R-bounded we fix some angle $\frac{\pi}{2} < \varphi < \pi$. Then for any choice of numbers $\zeta_j = r_j e^{i\varphi_j}$ with $\varphi_j \geq \varphi$, $j = 1, \dots, n$ we obtain

$$\begin{aligned} (\mathbb{E} \|\sum_{j=1}^n \epsilon_j \zeta_j R(\zeta_j, A) x_j\|^2)^{\frac{1}{2}} &= (\mathbb{E} \|\sum_{j=1}^n \int_0^\infty \epsilon_j r_j e^{i\varphi_j} e^{tr_j e^{i\varphi_j}} e^{-tA} x_j dt\|^2)^{\frac{1}{2}} \\ &\leq (\int_0^\infty \mathbb{E} \|\sum_{j=1}^n \epsilon_j e^{i\varphi_j} e^{se^{i\varphi_j}} e^{-s/r_j A} x_j\|^2 ds)^{\frac{1}{2}} \\ &\leq C (\int_0^\infty \max_j |e^{se^{i\varphi_j}}|^2 ds)^{\frac{1}{2}} (\mathbb{E} \|\sum_{j=1}^n \epsilon_j x_j\|^2)^{\frac{1}{2}} \\ &\leq \frac{C}{|\cos \varphi|} (\mathbb{E} \|\sum_{j=1}^n \epsilon_j x_j\|^2)^{\frac{1}{2}}. \end{aligned}$$

Therefore A is R-sectorial with sectoriality angle $\omega_R(A) \leq \pi/2$. Similarly, it follows that if A is R-sectorial and $\omega_R(A) < \pi/2$ then $\{e^{-tA} : 0 < t < \infty\}$ is R-bounded.

For the last statement suppose that C is a constant such that

$$(\mathbb{E} \|\sum_{j=1}^n \epsilon_j e^{-t_j A} x_j\|^2)^{\frac{1}{2}} \leq C (\mathbb{E} \|\sum_{j=1}^n \epsilon_j x_j\|^2)^{\frac{1}{2}}$$

whenever $x_1, \dots, x_n \in X$, $0 \leq t_1, \dots, t_n \leq 1$.

Then if $m \in \mathbb{N}$,

$$(\mathbb{E} \|\sum_{j=1}^n \epsilon_j e^{-(m+t_j)A} x_j\|^2)^{\frac{1}{2}} \leq CK^m (\mathbb{E} \|\sum_{j=1}^n \epsilon_j x_j\|^2)^{\frac{1}{2}}$$

where $K = \|e^{-A}\|$. Now we show that the set $\{e^{-ut} e^{-tA} : 0 < t < \infty\}$ is R-bounded as long as $e^u > K$. For $x_1, \dots, x_n \in X$ and $0 < t_1, \dots, t_n < \infty$ we obtain

$$\begin{aligned} (\mathbb{E} \|\sum_{j=1}^n \epsilon_j e^{-t_j u} e^{-t_j A} x_j\|^2)^{\frac{1}{2}} &= (\mathbb{E} \|\sum_{m=0}^\infty \sum_{m \leq t_j < m+1} \epsilon_j e^{-t_j u} e^{-t_j A} x_j\|^2)^{\frac{1}{2}} \\ &\leq C \sum_{m=0}^\infty K^m e^{-um} (\mathbb{E} \|\sum_{m \leq t_j < m+1} \epsilon_j e^{-u \tilde{t}_j} x_j\|^2)^{\frac{1}{2}} \end{aligned}$$

where $0 \leq \tilde{t}_j \leq 1$. By the contraction principle

$$(\mathbb{E} \|\sum_{m \leq t_j < m+1} \epsilon_j e^{-u \tilde{t}_j} x_j\|^2)^{\frac{1}{2}} \leq \max_{1 \leq j \leq n} |e^{-u \tilde{t}_j}| (\mathbb{E} \|\sum_{j=1}^n \epsilon_j x_j\|^2)^{\frac{1}{2}} \leq (\mathbb{E} \|\sum_{j=1}^n \epsilon_j x_j\|^2)^{\frac{1}{2}}$$

Since $\sum_{m=0}^\infty K^m e^{-um}$ is finite for $u > \ln K$ we obtain the claim. Consequently, the set $\{\xi R(\xi, u+A) : |\arg \xi| > \phi\}$ is R-bounded.

Now for $\zeta \in \mathbb{C}$ with $|\arg(\zeta + M)| > \phi$, $M > u$ we can rewrite using $\xi - u = \zeta$,

$$\zeta R(\zeta, A) = (\xi - u)R(\xi - u) = (\xi - u)R(\xi, A + u) = \frac{\xi - u}{\xi} \xi R(\xi, A + u)$$

Since $|\frac{\xi - u}{\xi}| \leq \frac{M}{M - u}$ the result follows quickly. \square

It was shown in [8] that an operator with H^∞ -calculus on L_1 is already R -sectorial. Moreover, there are no R -bounded strongly continuous semigroups on L_1 consisting of weakly compact operators [5], and if $-A$ generates a weakly compact semigroup (e^{-uA}) then the resolvents $R(\xi, A)$ are necessarily weakly compact [10]. Now, in the case where A is a sectorial operator which generates an R -bounded semigroup on L_1 , we will go even further and show that the resolvents have a representation quite unlike a weakly compact operator. Namely, they contain a multiple of the identity on L_1 .

Theorem 3.2. *Suppose A is a sectorial operator on $L_1(K, \lambda)$. Assume that either:*

- (i) *A is R -sectorial for some angle ω ,*
 or (ii) *$-A$ is generator of a bounded semigroup such that $\{e^{-tA} : 0 < t \leq 1\}$ is R -bounded.*

Then there is a bounded function $a(\zeta, s)$ defined for $s \in K$ and $|\arg \zeta| > \omega$ such that

- *For each $s \in K$ the map $\zeta \rightarrow a(\zeta, s)$ is analytic.*
- *For each ζ the map $s \rightarrow a(\zeta, s)$ is Borel.*
- *$\lambda\{s : a(\zeta, s) = 0\} = 0$ for almost every ζ .*
-

$$R(\zeta, A)f = a(\zeta, s)f(s) + \int_K f(t)d\mu_s^\zeta(t) \quad f \in L_1$$

where $\mu_s^\zeta\{s\} = 0$.

Proof. We begin with the observation that, under either hypothesis, there exists $\phi < \pi$ and $M < \infty$ such that the set of operators $\{\zeta R(\zeta, A) : |\arg \phi| \geq \phi, |\zeta| \geq M\}$ is R -bounded. Hence again the set of measures $\{\rho_{R(\zeta, A)} : |\arg \phi| \geq \phi, |\zeta| \geq M\}$ is relatively weakly compact.

Consider the map $\zeta \rightarrow \Pi(R(\zeta, A))$ which is an analytic map from the set $\mathcal{S} = \{\zeta : |\arg \zeta| > \omega\}$ into $\mathcal{L}(L_1)$. This induces an analytic map $F : \mathcal{S} \rightarrow L_\infty(K, \lambda)$ given by

$$\Pi(R(\zeta, A))f = F(\zeta)f.$$

Let us show that we can choose representatives so that $F(\zeta)(s) = a(\zeta, s)$ where a satisfies the first two conditions of the statement. Indeed let \mathbb{D} be the unit disk and let $\varphi : \mathbb{D} \rightarrow \mathcal{S}$ be a conformal equivalence. Then $F \circ \varphi$ can be expanded in a Taylor series around the origin and we may pick uniformly bounded Borel representatives b_n for the coefficients in the expansion so that

$$F(\varphi(z))(s) = \sum_{n=0}^{\infty} b_n(s)z^n \quad \lambda - \text{a.e.}$$

Let

$$a(\zeta, s) = \sum_{n=0}^{\infty} b_n(s)(\varphi^{-1}(\zeta))^n.$$

Assume that the third condition fails. Then by Fubini's theorem there is a subset B of K with $\lambda(B) > 0$ so that for each $s \in B$ the set $\{\zeta : a(\zeta, s) = 0\}$ has positive planar measure. By analyticity, this implies $a(\zeta, s) \equiv 0$ for $s \in B$.

However $\rho_{n(n+A)^{-1}}$ converges weakly to ρ_I and hence so does $\rho_{\Pi(n(n+A)^{-1})}$. Thus $-na(-n, s)$ is weakly convergent to the constantly one function in $L_\infty(K, \lambda)$. This is a contradiction. \square

The next theorem shows that if a sectorial operator generates an R-bounded semigroup on L_1 then it is very similar to a bounded operator in the sense that its domain is sufficiently large to contain generic L_1 -functions.

Theorem 3.3. *Let A be a sectorial operator on L_1 and assume that for some $\phi < \pi$ and $M < \infty$ the set $\{\zeta R(\zeta, A) : |\arg \zeta| \geq \phi, |\zeta| \geq M\}$ is R-bounded. Then for any $\epsilon > 0$ there is an invertible operator $U : L_1 \rightarrow L_1$ with $\|U - I\| < \epsilon$ and a density function $w > 0$ a.e. such that $L_1(w) \subset U^{-1}(\mathcal{D}(A))$.*

Proof. According to Proposition 2.3 the set of measures $\rho_{\zeta R(\zeta, A)}$ for $|\arg \zeta| \geq \phi, |\zeta| \geq M$ is relatively weakly compact in $\mathcal{M}(K \times K)$. The sequence $(n(n+A)^{-1})_{n \geq M}$ converges in the strong operator topology to the identity. Therefore, $\rho_{n(n+A)^{-1}}$ converges *weak** to ρ_I in $\mathcal{M}(K \times K)$ and hence converges weakly to ρ_I by weak compactness.

Now by Proposition 2.1 we may find a sequence of convex combinations $(T_n)_{n=1}^\infty$ of $(n(n+A)^{-1})_{n=1}^\infty$, and a sequence of Borel sets $E_n \subset K$ such that $\lambda(E_n) > 1 - 2^{-n}\epsilon$ and

$$\|T_n f - f\| \leq 2^{-n}\epsilon \|f\| \quad f \in L_1(E_n).$$

Let us put $F_1 = E_1$ and then $F_n = E_n \setminus E_{n-1}$ for $n \geq 2$. We define $U : L_1 \rightarrow L_1$ by

$$Uf = \sum_{n=1}^{\infty} T_n(f\chi_{F_n}).$$

Thus $\|U - I\| \leq \epsilon$. Observe that $T_n : L_1 \rightarrow \mathcal{D}(A)$ and so AT_n is a bounded operator on L_1 . Define

$$w = \sum_{n=1}^{\infty} \|AT_n\| \chi_{F_n}$$

and assume $f \in L_1(w)$. Then

$$\|AU(f\chi_{F_n})\| = \|AT_n(f\chi_{F_n})\| \leq \int_{F_n} |f|w \, dt.$$

Hence $\sum_{k=1}^{\infty} AU(f\chi_{F_k})$ converges and, since A is closed, $Uf \in \mathcal{D}(A)$. \square

Typically, differential operators on finite domains have compact resolvents. Therefore, they do not generate R-bounded semigroups [5]. In contrast, resolvents of differential operators on infinite domains are generally not compact. An important example is the Laplacian Δ on $L_1(\mathbb{R}^n)$. It does not have a compact resolvent and is hence not covered by the theorem in [5]. Our corollary addresses this situation.

Corollary 3.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with locally Lipschitz boundary or $\Omega = \mathbb{R}^n$. Suppose that $A : L_1(\Omega) \supset \mathcal{D}(A) \rightarrow L_1(\Omega)$ is a sectorial operator such that $\mathcal{D}(A)$ is contained in a Sobolev space $H_1^s(\Omega)$ for some $s > 0$. Then A does not generate an R-bounded semigroup.*

Proof. Assume the contrary, i.e. A generates an R-bounded semigroup. Then by Sobolev's embedding theorem [1] we have a continuous inclusion $H_1^s(\Omega) \hookrightarrow L_p(\Omega)$ for some $p > 1$. Using Theorem 3.3 we obtain that there is a positive weight function w such that $L_1(\Omega, w)$ is isomorphically embedded in $\mathcal{D}(A)$ and thus in $L_p(\Omega)$. This is impossible. \square

This corollary is actually true for any set Ω for which Sobolev's embedding theorem holds. Sufficient geometrical properties of Ω for this to happen are discussed in [1].

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