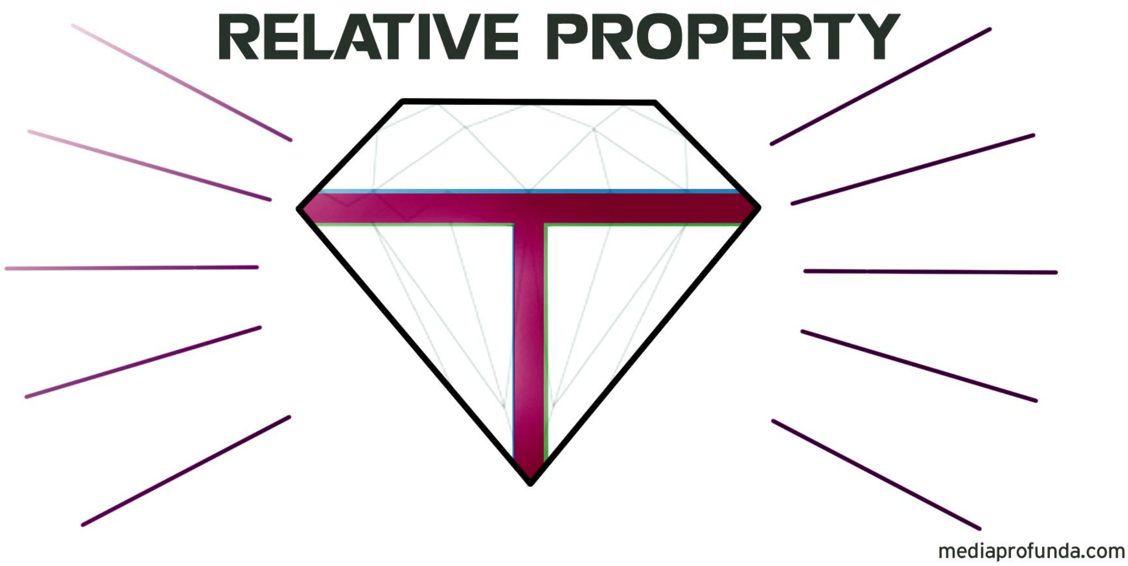
New Examples of Group Pairs

with Kazhdan's



Talia Fernós

Outline of the Talk

- Definitions.
- Examples of group pairs **without** relative property (T).
- Examples of group pairs **with** relative property (T).
- Applications and Observations.
- Statement of the main result.
- Discussion of the main tools.
- Proof of a special case.

Definitions

Definition 1. *A unitary representation* $\pi\colon G \to \mathcal{U}(\mathcal{H})$ *is said to have* almost invariant vectors *if there exists a sequence of unit vectors* $u_n \in \mathcal{H}$ *such that for every* $K \subset G$ *compact*

$$\max_{k \in K} \|\pi(k)u_n - u_n\| \to 0.$$

Definition 2. *Let* $A \leqslant G$. *The group pair* (G, A) *is said to have* relative property (T) *if every unitary representation* $\pi\colon G \to \mathcal{U}(\mathcal{H})$ *with almost invariant vectors has* A*-invariant vectors.*

Definition 3. *A group* G *is said to have* property (T) *if* (G, G) *has relative property (T).*

Non-Example: Suppose that A and S are amenable groups and S acts on A by automorphisms. Then $(S \ltimes A, A)$ has relative property (T) if and only A is compact.

- The right regular representation

$$\rho\colon S \ltimes A \to L^2(S \ltimes A, Haar)$$

has almost invariant vectors since $S \ltimes A$ is amenable.

- A non-zero A-invariant function is constant on the left cosets of A.

- Since such a function is square integrable, it follows that A is compact.

Examples of discrete group pairs with relative property (T):

- [Wang 1975] If $n \geqslant 2$ then

$$(\mathsf{SL}_n(\mathbb{Z}) \ltimes \mathbb{Z}^n, \mathbb{Z}^n)$$

has relative property (T).

- [Burger 1991] If $\Gamma \leqslant \mathsf{SL}_2(\mathbb{Z})$ is non-amenable then $(\Gamma \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ has relative property (T).

- [Valette 2004] If Γ is an arithmetic lattice in an absolutely simple Lie group (with trivial center) then there exists a homomorphism $\varphi\colon \Gamma \to \mathsf{SL}_n(\mathbb{Z})$ such that

$$(\Gamma \ltimes_\varphi \mathbb{Z}^n, \mathbb{Z}^n)$$

has relative property (T).

Recent Applications:

- [Popa 2004] Discrete group pairs with relative property (T) of the form $(\Gamma \ltimes A, A)$, where A is an Abelian group, give rise (under additional hypotheses) to II_1 factors with rigid Cartan subalgebra inclusion.

- [Navas 2004] Let $\Gamma \leqslant C^2(S^1)$ be a countable group. If $A \leqslant \Gamma$ such that (Γ, A) has relative property (T) then A is finite.

Observations:

- There is a renewed interest in relative property (T).

- Until Valette's recent result, there were few examples of group pairs with relative property (T).

Question: To what extent can group pairs with relative property (T) be constructed?

Theorem (Fernós). *Let* Γ *be a finitely generated group. If there exists a homomorphism* $\varphi\colon \Gamma \to \mathrm{SL}_n(\mathbb{R})$ *such that the Zariski closure* $\overline{\varphi(\Gamma)}^Z(\mathbb{R})$ *is non-amenable, then there exists an Abelian group* A *of finite* $\mathbb{Q}$*-rank such that* Γ *acts on* A *by automorphisms and the corresponding group pair* $(\Gamma \ltimes A, A)$ *has relative property (T).*

Our proof is based on Burger's examples mentioned above. We now

- ❖ discuss the two main tools of the proof (Lemmas 1 and 2).
- ❖ Prove a special case to illustrate the use of these tools.

$\mathbb{K}$ = a non-discrete local field (e.g. $\mathbb{K} = \mathbb{R}$).

Lemma 1 (Burger's Criterion). *Suppose* Γ *is a group and* $\Gamma \to \mathrm{GL}_N(\mathbb{K})$ *a homomorphism. If there are no* Γ*-invariant probability measures on* $\mathbb{P}(\mathbb{K}^N)$. *Then* $(\Gamma \ltimes \mathbb{K}^N, \mathbb{K}^N)$ *has relative property (T).*

- The Spectral Theorem translates between unitary representations and measures on $\mathbb{K}^N$.

Proof. Contrapositive:

If $(\Gamma \ltimes \mathbb{K}^N, \mathbb{K}^N)$ **does not** have relative property (T) then there exists a $\pi \colon \Gamma \ltimes \mathbb{K}^N \to \mathcal{U}(\mathcal{H})$ with

- almost invariant vectors $u_n \in \mathcal{H}$

- no $\mathbb{K}^N$-invariant vectors.

The Spectral Theorem yields the existence of a projection valued measure

$$P\colon \text{Borel}(\mathbb{K}^N) \to \text{Proj}(\mathcal{H})$$

It has the following properties:

- $P(\mathbb{K}^N) =$ Identity Projection

- $P(\{0\}) = \left\{ \begin{array}{c} \text{projection onto the subspace} \\ \text{of } \mathbb{K}^N\text{-invariant} \end{array} \right.$

- By assumption $P(\{0\}) \equiv 0$.

- For each $\gamma \in \Gamma$

$$\pi(\gamma^{-1})P(B)\pi(\gamma) = P(\gamma B).$$

- The measure $\mu_n(B) := \langle P(B)u_n, u_n\rangle$ is a positive Borel probability measure supported on $\mathbb{K}^N \backslash \{0\}$.

Fact 1. *Since the vectors* $\{u_n\}$ *are almost* Γ*-invariant, the measures* $\{\mu_n\}$ *are almost* Γ*-invariant. (Uses the* Γ*-equivariance of the projection valued measure.)*

Fact 2. *If* $p\colon \mathbb{K}^N \backslash \{0\} \to \mathbb{P}(\mathbb{K}^N)$ *is the natural projection then the push forwards* $\{p_*\mu_n\}$ *are almost* Γ*-invariant probability measures on* $\mathbb{P}(\mathbb{K}^N)$*.*

Fact 3. *A weak-*$\star$ *accumulation point of* $\{p_*\mu_n\}$ *is a* Γ*-invariant probability measure on* $\mathbb{P}(\mathbb{K}^N)$*.*

We have shown:
If $(\Gamma \ltimes \mathbb{K}^N, \mathbb{K}^N)$ does not have relative property (T) then there is a Γ-invariant probability measure on $\mathbb{P}(\mathbb{K}^N)$. ✿

Lemma 2 (Furstenberg's Lemma). *Suppose that* $\Gamma \leqslant \mathsf{PGL}_N(\mathbb{K})$ *is not precompact and* μ *is a* Γ*-invariant probability measure on* $\mathbb{P}(\mathbb{K}^N)$*. Then there exists a nonzero subspace* $V \subsetneq \mathbb{K}^N$ *which is virtually* Γ*-invariant and such that* $\mu[V] > 0$*.*

- Makes use of the fact that the Grassmann varieties (i.e. the projective space of k-planes in $\mathbb{K}^N$) are compact.

Proposition (Fernós). *Let* $\Gamma \leqslant \mathsf{SL}_N(\mathbb{Z})$ *be a non-amenable subgroup such that*

1. Γ *has no eigenvectors in* $\mathbb{R}^N$

2. *the Zariski-closure* $\overline{\Gamma}^Z(\mathbb{R})$ *is connected and has no compact quotients.*

Then $(\Gamma \ltimes \mathbb{Z}^N, \mathbb{Z}^N)$ *has relative property (T).*

The proof is in three steps:

1. Observe that $\Gamma \ltimes \mathbb{Z}^N \leqslant \Gamma \ltimes \mathbb{R}^N$ is a lattice.

2. Observe that "lattices inherit relative property (T)". This follows by **induction**:

If $\pi\colon \Gamma \ltimes \mathbb{Z}^N \to \mathcal{U}(\mathcal{H})$ is a unitary representation, then there exists
$\tilde{\pi}\colon \Gamma \ltimes \mathbb{R}^N \to \mathcal{U}(\tilde{\mathcal{H}})$ such that

- **If** π has almost invariant vectors **then** $\tilde{\pi}$ has almost invariant vectors.
- **If** $\tilde{\pi}$ has $\mathbb{R}^N$-invariant vectors **then** π has $\mathbb{Z}^N$-invariant vectors.

3. Show that $(\Gamma \ltimes \mathbb{R}^N, \mathbb{R}^N)$ has relative property (T) using Burger's Criterion.

Proof of Step 3. By contradiction:

If $(\Gamma \ltimes \mathbb{R}^N, \mathbb{R}^N)$ **does not** have relative property (T) then

- By **Burger's Criterion** there is a measure μ on $\mathbb{P}(\mathbb{R}^N)$ that is Γ-invariant.

- By **Furstenberg's Lemma**

  - there is a non-zero subspace $V_0 \subsetneq \mathbb{R}^N$ such that $\mu[V_0] > 0$.

  - there is a finite index subgroup $\Gamma_0 \leqslant \Gamma$ such that $\Gamma_0 \cdot V_0 = V_0$.

- V_0 is **minimal** among all such subspaces.

- Consider the Γ_0-invariant measure:

$$\mu_0(B) = \frac{1}{\mu[V_0]}\mu(B \cap [V_0])$$

- By **Furstenberg's** Lemma and **minimality** of V_0 the image of

$$\rho\colon \Gamma_0 \to \mathrm{PGL}(V_0)$$

is **pre-compact**.

- We will now see that this contradicts our assumptions.

Some Facts:

- If Γ_0 preserves V_0 then so does $H_0 = \overline{\Gamma_0}^Z(\mathbb{R})$.
- If the index $[\Gamma : \Gamma_0] < \infty$ and $H = \overline{\Gamma}^Z(\mathbb{R})$ is connected then $H = H_0$.
- If $\rho(\Gamma_0)$ is precompact then $\rho(H)$ is compact.

But H **does not have** compact quotients.

So $\rho(H) = 1$.

Therefore every vector in V_0 is an **eigenvector** of H and hence of Γ.

This **contradicts** our assumption that Γ has no eigenvectors. ✿

Before turning to our main theorem some considerations:

- $\mathbb{Z}\left[\frac{1}{p}\right] \stackrel{\text{def}}{=} \left\{\frac{a}{p^k} \colon a \in \mathbb{Z} \text{ and } k \in \mathbb{N}\right\}$.

- $\mathbb{Q}_p \stackrel{\text{def}}{=} \left\{\sum_{i=-k}^{\infty} a_i p^i \colon a_i \in \{0, 1, \ldots, p-1\}\right\}$.

- There is a natural norm on $\mathbb{Q}_p$ called the p-adic norm where $\|p^k\| = p^{-k}$.

- The field $\mathbb{Q}_p$ is locally compact and non-discrete with the p-adic norm topology.

- The diagonal embedding $\mathbb{Z}\left[\frac{1}{p}\right] \hookrightarrow \mathbb{R} \times \mathbb{Q}_p$ is discrete and co-compact.

- More generally, if S is a finite set of primes then $\mathbb{Z}[S^{-1}]$ is the set of rationals where we are allowed to divide by primes in S.

- The diagonal embedding

$$\mathbb{Z}[S^{-1}] \hookrightarrow \mathbb{R} \times \prod_{p \in S} \mathbb{Q}_p$$

is discrete and co-compact.

- It is common to write $\mathbb{Q}_\infty \stackrel{\text{def}}{=} \mathbb{R}$.

Keep in mind these examples:

- If S is empty then $\mathbb{Z}[S^{-1}] = \mathbb{Z}$.

- If $S = \{p\}$ then $\mathbb{Z}[S^{-1}] = \mathbb{Z}\left[\frac{1}{p}\right]$.

Theorem (Fernós). *Let* Γ *be a finitely generated group. If there exists a homomorphism* $\varphi\colon \Gamma \to \mathrm{SL}_n(\mathbb{R})$ *such that the Zariski closure* $\overline{\varphi(\Gamma)}^Z(\mathbb{R})$ *is non-amenable, then there exists*

- *a finite set of primes* S *and*

- *a homomorphism* $\alpha\colon \Gamma \to \mathrm{SL}_N(\mathbb{Z}[S^{-1}])$

such that $(\Gamma \ltimes_\alpha \mathbb{Z}[S^{-1}]^N, \mathbb{Z}[S^{-1}]^N)$ *has relative property (T).*

Step 1 (From Transcendental to Arithmetic). *Let* Γ *be a finitely generated group. If there exists a homomorphism* $\varphi\colon \Gamma \to \mathsf{SL}_n(\mathbb{R})$ *such that the Zariski-closure* $\overline{\varphi(\Gamma)}^Z(\mathbb{R})$ *is non-amenable then there exists a homomorphism*

$$\psi\colon \Gamma \to \mathsf{SL}_m(\mathbb{Q})$$

such that $\overline{\psi(\Gamma)}^Z(\mathbb{R})$ *is non-amenable.*

Remark: A $\mathbb{R}$-algebraic group is amenable if and only if it is a compact extension of its solvable radical.

Tools:

- Restriction of scalars functor

- Specialization of rings that are finitely generated over $\mathbb{Q}$.

- Finding a finite index normal subgroup $\Gamma_n \trianglelefteq \Gamma$ such that for any $\varphi\colon \Gamma \to \mathsf{SL}_n(\mathbb{R})$ TFAE:

  1. The Zariski-closure $\overline{\varphi(\Gamma)}^Z(\mathbb{R})$ is amenable.

  2. The traces of the commutator subgroup $tr(\varphi([\Gamma_n, \Gamma_n]))$ are uniformly bounded.

Step 2 (Relative property (T) with Local Fields). *If* Γ *is a finitely generated group satisfying property* (F_p) *then there exists a homomorphism* $\Phi_p\colon \Gamma \to \mathrm{SL}_N(\mathbb{Q})$ *such that*

$$(\Gamma \ltimes_{\Phi_p} \mathbb{Q}_p^N, \mathbb{Q}_p^N)$$

has relative property (T).

- Property (F_p) (after Furstenberg):

  The existence of an arithmetic representation (into $\mathrm{SL}_N(\mathbb{Q})$) satisfying strong irreducibility and $\mathbb{Q}_p$-unboundedness conditions.

Step 3 (Fixing the Primes). *If* Γ *is finitely generated and has property* (F_∞) *then there exists a finite set of primes* S *and a homomorphism* $\alpha\colon \Gamma \to \mathsf{SL}_N(\mathbb{Z}[S^{-1}])$ *such that for each* $p \in S \cup \{\infty\}$ *the corresponding group pair*

$$(\Gamma \ltimes_\alpha \mathbb{Q}_p^N, \mathbb{Q}_p^N)$$

has relative property (T).

- The homomorphism α above is a conjugate of Φ_∞ arising from property (F_∞).

- Step 1 (the existence of $\Gamma \to \mathsf{SL}_m(\mathbb{Q})$ with nonamenable Zariski closure) implies that Γ satisfies property (F_∞).

Step 4 (Products). *If* $(\Gamma \ltimes \mathbb{Q}_p^N, \mathbb{Q}_p^N)$ *has relative property (T) for each* $p \in S \cup \{\infty\}$ *then setting* $V = \prod_{p \in S \cup \{\infty\}} \mathbb{Q}_p^N$ *we have that* $(\Gamma \ltimes V, V)$ *has relative property (T).*

Step 5 (Induction). *Let* $A = \mathbb{Z}[S^{-1}]^N$*. Since* $A \leqslant V$ *is a lattice, and* $V \trianglelefteq \Gamma \ltimes V$ *we have that* $(\Gamma \ltimes A, A)$ *has relative property (T).*

We have shown:

Theorem. *Let* Γ *be a finitely generated group. If there exists a homomorphism* $\varphi\colon \Gamma \to \mathsf{SL}_n(\mathbb{R})$ *such that the Zariski closure* $\overline{\varphi(\Gamma)}^{Z}(\mathbb{R})$ *is non-amenable, then there exists an Abelian group* A *of finite* $\mathbb{Q}$*-rank such that* Γ *acts on* A *by automorphisms and the corresponding group pair* $(\Gamma \ltimes A, A)$ *has relative property (T).*