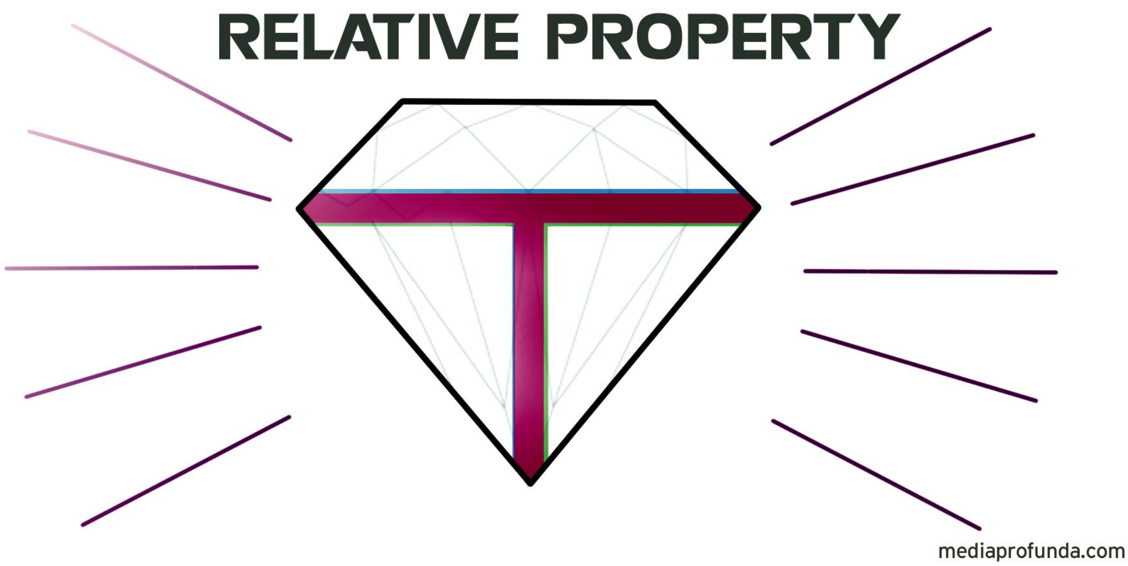
New Examples of Group Pairs

with Kazhdan's



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Definitions

Definition 1. *A unitary representation* $\pi\colon G \to \mathcal{U}(\mathcal{H})$ *is said to have* almost invariant vectors *if there exists a sequence of unit vectors* $u_n \in \mathcal{H}$ *such that for every* $K \subset G$ *compact*

$$\max_{k \in K} \|\pi(k)u_n - u_n\| \to 0.$$

Definition 2. *Let* $A \leqslant G$. *The group pair* (G, A) *is said to have* relative property (T) *if every unitary representation* $\pi\colon G \to \mathcal{U}(\mathcal{H})$ *with almost invariant vectors has* A*-invariant vectors.*

Definition 3. *A group* G *is said to have* property (T) *if* (G, G) *has relative property (T).*

Examples of discrete group pairs with relative property (T):

- [Wang 1975] If $n \geqslant 2$ then

$$(\mathsf{SL}_n(\mathbb{Z}) \ltimes \mathbb{Z}^n, \mathbb{Z}^n)$$

has relative property (T).

- [Burger 1991] If $\Gamma \leqslant \mathsf{SL}_2(\mathbb{Z})$ is non-amenable then $(\Gamma \ltimes \mathbb{Z}^2, \mathbb{Z}^2)$ has relative property (T).

- [Valette 2004] If Γ is an arithmetic lattice in an absolutely simple Lie group (with trivial center) then there exists a homomorphism $\varphi\colon \Gamma \to \mathsf{SL}_n(\mathbb{Z})$ such that

$$(\Gamma \ltimes_\varphi \mathbb{Z}^n, \mathbb{Z}^n)$$

has relative property (T).

Non-Example: Suppose that A and S are countable amenable groups and S acts on A by automorphisms. Then $(S \ltimes A, A)$ has relative property (T) if and only A is finite.

- The right regular representation

  $$\rho\colon S \ltimes A \to \ell^2(S \ltimes A)$$

  has almost invariant vectors since $S \ltimes A$ is amenable.

- A non-zero A-invariant function is constant on the left cosets of A.

- Since such a function is square summable, it follows that A is finite.

Recent Applications:

- [Navas 2004] Let $\Gamma \leqslant C^2(S^1)$ be a countable group. If $A \leqslant \Gamma$ such that (Γ, A) has relative property (T) then A is finite.

- [Popa 2004] Discrete group pairs with relative property (T) of the form $(\Gamma \ltimes A, A)$, where A is an Abelian group, give rise (under additional hypotheses) to II_1 factors with rigid Cartan subalgebra inclusion.

Observations:

- There is a renewed interest in relative property (T).

- Until Valette's recent result, there were few examples of group pairs with relative property (T).

Question: To what extent can group pairs with relative property (T) be constructed?

Theorem (Fernós). *Let* Γ *be a finitely generated group. If there exists a homomorphism* $\varphi\colon \Gamma \to \mathrm{SL}_n(\mathbb{R})$ *such that the Zariski closure* $\overline{\varphi(\Gamma)}^Z(\mathbb{R})$ *is non-amenable, then there exists an Abelian group* A *of finite* $\mathbb{Q}$*-rank such that* Γ *acts on* A *by automorphisms and the corresponding group pair* $(\Gamma \ltimes A, A)$ *has relative property (T).*

Remark: Actually, we provide more information:

$$A = \mathbb{Z}[S^{-1}]^N$$

where S is a finite set of primes. Where

$$\mathbb{Z}[S^{-1}] = \left\{\frac{a}{b} \in \mathbb{Q}\colon (a,b) = 1 \text{ and } p|b \text{ iff } p \in S\right\}.$$

Strategy of Proof:

- Observe that $A = \mathbb{Z}[S^{-1}]^N$ is a lattice in $V = \prod_{p \in S \cup \{\infty\}} \mathbb{Q}_p^N$. (Where $\mathbb{Q}_\infty := \mathbb{R}$.)

- If $\Gamma \to \mathsf{SL}_N(\mathbb{Z}[S^{-1}])$ is such that $(\Gamma \ltimes V, V)$ has relative property (T) then $(\Gamma \ltimes A, A)$ has relative property (T).

Step 1 (From Transcendental to Arithmetic). *Let Γ be a finitely generated group. If there exists a homomorphism $\varphi\colon \Gamma \to \mathsf{SL}_n(\mathbb{R})$ such that the Zariski-closure $\overline{\varphi(\Gamma)}^Z(\mathbb{R})$ is non-amenable then there exists a homomorphism*

$$\psi\colon \Gamma \to \mathsf{SL}_m(\mathbb{Q})$$

such that $\overline{\psi(\Gamma)}^Z(\mathbb{R})$ is non-amenable.

Remark: A $\mathbb{R}$-algebraic group is amenable if and only if it is a compact extension of its solvable radical.

Tools:

- Restriction of scalars functor

- Specialization of rings that are finitely generated over $\mathbb{Q}$.

- Finding a finite index normal subgroup $\Gamma_n \trianglelefteq \Gamma$ such that for any $\varphi \colon \Gamma \to \mathsf{SL}_n(\mathbb{R})$ TFAE:

  1. The Zariski-closure $\overline{\varphi(\Gamma)}^Z(\mathbb{R})$ is amenable.

  2. The traces of the commutator subgroup *tr*$(\varphi([\Gamma_n, \Gamma_n]))$ are uniformly bounded.

Step 2 (Relative Property (T) with Local Fields). *If* Γ *is a finitely generated group satisfying property* (F_p) *then there exists a homomorphism* $\Phi_p \colon \Gamma \to \mathsf{SL}_N(\mathbb{Q})$ *such that*

$$(\Gamma \ltimes_{\Phi_p} \mathbb{Q}_p^N, \mathbb{Q}_p^N)$$

has relative property (T).

- Property (F_p) (after Furstenberg):

  The existence of an arithmetic representation (into $\mathsf{SL}_N(\mathbb{Q})$) satisfying strong irreducibility and $\mathbb{Q}_p$-unboundedness conditions.

Lemma 1 (Burger's Criterion). *Suppose* Γ *is a group and* $\Gamma \to \mathsf{GL}_N(\mathbb{Q}_p)$ *a homomorphism. If there are no* Γ*-invariant probability measures on* $\mathbb{P}(\mathbb{Q}_p^N)$*. Then* $(\Gamma \ltimes \mathbb{Q}_p^N, \mathbb{Q}_p^N)$ *has relative property (T).*

- The Spectral Theorem is key: It provides a dictionary between unitary representations and measures on $\mathbb{Q}_p^N$.

Lemma 2 (Furstenberg's Lemma). *Suppose that* $\Gamma \leqslant \mathsf{PGL}_N(\mathbb{Q}_p)$ *is not precompact and* μ *is a* Γ*-invariant probability measure on* $\mathbb{P}(\mathbb{Q}_p^N)$*. Then there exists a nonzero subspace* $V \subsetneq \mathbb{Q}_p^N$ *which is virtually* Γ*-invariant and such that* $\mu[V] > 0$*.*

- Makes use of the fact that the Grassmann varieties (i.e. the projective space of k-planes in $\mathbb{Q}_p^N$) are compact.

Step 3 (Fixing the Primes). *If Γ is finitely generated and has property (F_∞) then there exists a finite set of primes S and a homomorphism $\Gamma \to \mathrm{SL}_N(\mathbb{Z}[S^{-1}])$ such that for each $p \in S \cup \{\infty\}$ the corresponding group pair*

$$(\Gamma \ltimes \mathbb{Q}_p^N, \mathbb{Q}_p^N)$$

has relative property (T).

- The homomorphism $\Gamma \to \mathrm{SL}_N(\mathbb{Z}[S^{-1}])$ above is a conjugate of the one arising from property (F_∞).

- If Γ is a finitely generated group and there is a homomorphism $\Gamma \to \mathrm{SL}_m(\mathbb{Q})$ with non-amenable Zariski closure, then Γ satisfies property (F_∞).

Step 4 (Products). *If* $(\Gamma \ltimes \mathbb{Q}_p^N, \mathbb{Q}_p^N)$ *has relative property (T) for each* $p \in S \cup \{\infty\}$ *then setting* $V = \prod_{p \in S \cup \{\infty\}} \mathbb{Q}_p^N$ *we have that* $(\Gamma \ltimes V, V)$ *has relative property (T).*

Step 5 (Induction). *Let* $A = \mathbb{Z}[S^{-1}]^N$. *Since* $A \leqslant V$ *is a lattice, and* $V \trianglelefteq \Gamma \ltimes V$ *we have that* $(\Gamma \ltimes A, A)$ *has relative property (T).*

We have shown:

Theorem (Fernós). *Let* Γ *be a finitely generated group. If there exists a homomorphism* $\varphi\colon \Gamma \to \mathrm{SL}_n(\mathbb{R})$ *such that the Zariski closure* $\overline{\varphi(\Gamma)}^Z(\mathbb{R})$ *is non-amenable, then there exists an Abelian group* A *of finite* $\mathbb{Q}$*-rank such that* Γ *acts on* A *by automorphisms and the corresponding group pair* $(\Gamma \ltimes A, A)$ *has relative property (T).*

The proof of this theorem also yields many concrete examples of group pairs with relative property (T):

Example: If $\Gamma \leqslant \mathsf{SL}_N(\mathbb{Z})$ has no fixed vectors in $\mathbb{Q}^N$ and the Zariski closure $\overline{\Gamma}^Z(\mathbb{R})$ has no compact factors then $(\Gamma \ltimes \mathbb{Z}^N, \mathbb{Z}^N)$ has relative property (T).

This is almost a direct consequence of Burger's Criterion and Furstenberg's Lemma.