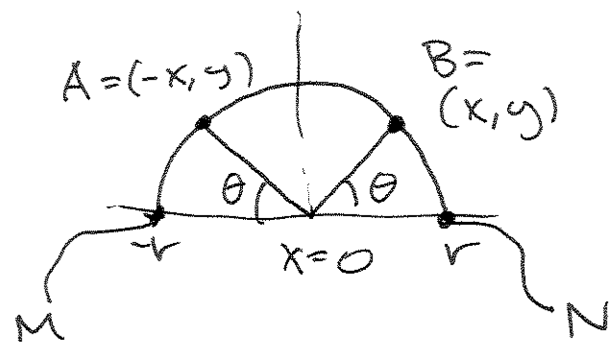
# ☆ Describing a hyperbolic circle :

Let's find points A & B so that:

* $d(A,B) = D$ &
*




SO: A & B are symmetric about the $x=0$ line.

$\therefore$ $x = r\cos\theta$ & $y = r\sin\theta$ .

To compute the hyperbolic distance $d(A,B)$ we need the Euclidean distances that appear in the cross ratio:

$$|AN| = \sqrt{(-x-r)^2+y^2} = \sqrt{x^2+2xr+r^2+y^2}$$
$$= \sqrt{2r^2+2xr} \quad (\text{b/c } x^2+y^2=r^2$$
$$= \sqrt{2r^2+2r^2\cos\theta} \quad (\text{b/c } x = r\cos$$
$$\boxed{= \sqrt{2r^2(1+\cos\theta)}} \quad ☆$$
$$= |BM| .$$

$$|AM| = \sqrt{(-x+r)^2+y^2}$$
$$= \sqrt{x^2-2xr+r^2+y^2} = \sqrt{2r^2-2r^2\cos\theta}$$
$$= \sqrt{2r^2(1-\cos\theta)} \quad ☆$$
$$= |BN|$$

Then: $d(A,B) = \left|\ln\left(\dfrac{AN\cdot BM}{AM\cdot BN}\right)\right|$

$= \left|\ln\left(\dfrac{2r^2(1+\cos\theta)}{2r^2(1-\cos\theta)}\right)\right|$

$= \left|\ln\left(\dfrac{1+\cos\theta}{1-\cos\theta}\right)\right|$ ★Note: This is independent of $r$!

$\overset{\text{By assump.}}{=} D > 0$

$\Rightarrow \dfrac{1+\cos\theta}{1-\cos\theta} = e^{\pm D}$

$\Rightarrow 1+\cos\theta = e^{\pm D} - e^{\pm D}\cos\theta$

$\Rightarrow \cos\theta(1+e^{\pm D}) = e^{\pm D} - 1$

$\Rightarrow \cos\theta = \dfrac{e^{\pm D}-1}{e^{\pm D}+1}$ ★Note: This gives 2-soluti[ons] corresponding to $x>$[0] & $x<0$; each negati[ve] of the other.

For us: $x = r\cos\theta > 0$ (see picture)

$\therefore \cos\theta = \dfrac{e^D-1}{e^D+1}$ Let $\theta=\theta_D$ be the ∡ between $[0,\pi/2]$ satisfying

Next: we need the y-coordinate.

Method 1: find $\theta_D$ & then $\sin\theta_D$. (hard)

Method 2: use $\sin\theta = +\sqrt{1-\cos^2\theta}$

← positive b/c pts $\in \mathbb{H}^2$

$$\Rightarrow \sin\theta_D = \sqrt{1 - \frac{(e^D-1)^2}{(e^D+1)^2}}$$

$$= \sqrt{\frac{(e^D+1)^2 - (e^D-1)^2}{(e^D+1)^2}}$$

← numerator = difference of squares

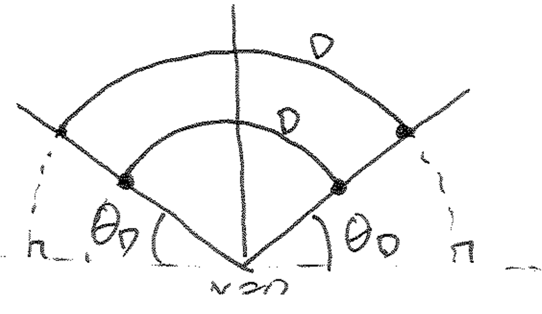
$$= \sqrt{\frac{(e^D+1-e^D+1)(e^D+1+e^D-1)}{(e^D+1)^2}}$$

$$= \sqrt{\frac{4e^D}{(e^D+1)^2}} = 2\,\frac{e^{D/2}}{e^D+1}.$$

$$\therefore\ B = (x,y) = \left(r\,\frac{e^D-1}{e^D+1},\ 2r\,\frac{e^{D/2}}{e^D+1}\right)$$

$$\&\ A = (-x,y) = \left(-r\,\frac{e^D-1}{e^D+1},\ 2r\,\frac{e^{D/2}}{e^D+1}\right).$$

Conclusion: For $D>0$ & $\theta_D$ as above:




pts with same Euclid vertical component along these Euclidean rays are all hyperbolic distance D apart.

Previously, we established:

<u>Lemma</u>: $f: \mathbb{R}^2 \to \mathbb{R}^2$ satisfying

1) $f$ = euclidean isometry; 2) $f(0,x) \overset{\text{preserves "x-axis"}}{=} f(0,x')$

3) $f: \mathbb{H}^2 \to \mathbb{H}^2$ (i.e. the "y"-coordinate remains positive in the image).

Such an $f$ is also a hyperbolic isometry

<u>Examples of such an $f$</u>:

* translation along x-axis

$$\tau_a(x,y) = (x+a, y)$$

* reflections along euclidean-vertical lines:

$$\rho_0(x,y) = (-x, y) = \text{reflection @ } x=0$$

$$\rho_a(x,y) = (-x+2a, y) = \text{reflection @ } x=a$$

$$= \tau_a \rho_0 \tau_a^{-1}(x,y)$$

Now, we will apply what we found above to get more points on $\mathcal{C}_1(0,1)$ = circle of radius 1 @ $(0,1)$.

[ <u>Fact</u>: The above list is complete; i.e: If $f$ = euclidean isom. satisfying conditions of lemma $\Rightarrow f = \tau_a$ or $\rho_a$ some $a$. ]

Let's take $D=1$ So:

$$A=\left(-r\,\frac{e-1}{e+1},\,2r\,\frac{e^{1/2}}{e+1}\right);\ B=\left(r\,\frac{e-1}{e+1},\,2r\,\frac{e^{1/2}}{e+1}\right)$$

are distance (hyp.) 1 from each other.

Now, we would like to make $A \rightarrow (0,1)$

we need an isometry that will do $\nearrow$.

<u>First</u>: pick $r$ so that $2r\frac{e^{1/2}}{e+1}=1$

i.e: $r=\frac{e+1}{2\sqrt{e}}$ (can do because $\theta_1$ is independent of $r$!

$$\Rightarrow A_1=\left(-\frac{e+1}{2\sqrt{e}}\cdot\frac{e-1}{e+1},\,2\,\frac{e+1}{2\sqrt{e}}\,\frac{\sqrt{e}}{e+1}\right)=\left(-\frac{e-1}{2\sqrt{e}},1\right)$$

$$\& \ B_1=\left(\frac{e-1}{2\sqrt{e}},1\right) \quad \therefore d(A_1,B_1)=1$$

$\therefore$ we would like to shift by $a=\frac{e-1}{2\sqrt{e}}$

$$\tau_a(A_1)=(0,1) \quad \& \quad \tau_a(B_1)=\left(\frac{e-1}{\sqrt{e}},1\right)$$

$\&\ d(\tau_a(A_1),\tau_a(B_1))=d(A_1,B_1)=1$.

<u>We also have</u>: $d(\rho\circ\tau_a(A_1),\rho\circ\tau_a(B_1))=1$

$$=d\left((0,1),\left(-\frac{e-1}{\sqrt{e}},1\right)\right)$$

<u>So</u>: $\left(\pm\frac{e-1}{\sqrt{e}},1\right)\in C_1(0,1)$.

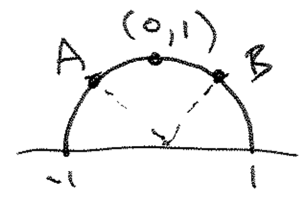
In fact: since $\rho_0(0,1)=(0,1)$ $\mathcal{C}_1(0,1)$ will be symmetric about the $x=0$ line

i.e. $(x,y)\in\mathcal{C}_1(0,1) \overset{\text{if \& only if}}{\Longleftrightarrow} (-x,y)\in\mathcal{C}_1(0,1)$.

Next: Consider points A & B:

belong to geodesic $\leftrightarrow$

& also belong to $\mathcal{C}_1(0,1)$




Observe: because $(0,1)$ lies on the geodesic between A & B & the geodesic is globally distance minimizing (unlike sphere

we get: $d(A,B) = d(A,(0,1)) + d((0,1),B)$

normally would be "$\leq$" but because lie on geodesic get $=$.

$\therefore d(A,B)=2$. Plugging into 1st part:

$$A=\left(-\frac{e^2-1}{e^2+1},\ 2\frac{e}{e^2+1}\right);\ B=\left(\frac{e^2-1}{e^2+1},\ \frac{2e}{e^2+1}\right)$$

Note: $r=1$ b/c. ~~circle~~ we working w/ Euclid. circle $r=1$

We had also found $(0,e^{\pm 1})\in\mathcal{C}_1(0,1)$ So:

(see next page)

Here is a plot of the points we've found on $\mathcal{C}_1(0,1)$

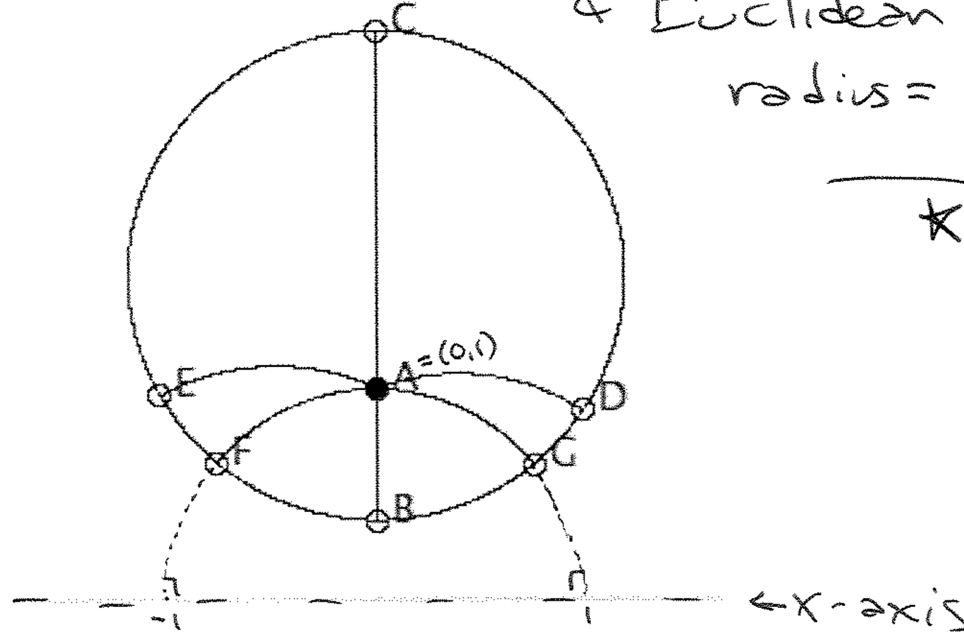
They all seem to lie on the Euclidean circle pictured below.

The equation of that circle is:

$$x^2+\left(y-\frac{e^2+1}{2e}\right)^2=\left(\frac{e^2-1}{2e}\right)^2$$

<u>Note</u>: The Euclidean center $=\left(0, \frac{e^2+1}{2e}\right)$

& Euclidean radius $=\frac{e^2-1}{2e}$




<u>In this picture</u>: $A=(0,1)$, $B=(0,e^{-1})$, $C=(0,e)$

$D=\left(\frac{e-1}{\sqrt{e}},1\right)$, $E=\left(-\frac{(e-1)}{\sqrt{e}},1\right)$

$F=\left(-\frac{e^2-1}{e^2+1}, 2\frac{e}{e^2+1}\right)$

$G=\left(\frac{e^2-1}{e^2+1}, 2\frac{e}{e^2+1}\right)$

$\mathbb{H}^2$ = "homogeneous" which means:

*) points are indistinguisable

i.e. $\forall P, P' \in \mathbb{H}^2$ $\exists f: \mathbb{H}^2 \to \mathbb{H}^2$ a hyp. <u>isometry</u> with $f(P) = P'$

**) directions are indistinguishable (horizon looks the same everywhere)

i.e. $\forall$ geodesics $\ell, \ell'$ passing throug $P \in \mathbb{H}^2$
(hyp. lines)

$\exists f: \mathbb{H}^2 \to \mathbb{H}^2$ a hyp. <u>isometry</u> with

$f(P) = P$ & $f(\ell) = \ell'$

* If $f$ preseves orientation (ie. ↻ maps to ↻)
$\Rightarrow$ can think of this $f$ as a rotation.

$\therefore$ There must be reflections which "swap" two sides determined by any geodesic, just as there is $\rho_0$.

Next we will consider $\rho_\ell(x,y) = \left( \frac{+x}{x^2+y^2}, \frac{y}{x^2+y^2} \right)$

<u>We will show</u>: $\rho_\ell$ has following properties

* Takes E-circles centered @ $(0,0)$ of radius $r$ to E-circles centered @ $(0,0)$ of radius $1/r$ } swaps "sides"

* maps $\mathbb{H}^2$ ↻, preserves many distances.