DIAGONAL SUPERCOMPACT RADIN FORCING

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Abstract. Motivated by the goal of constructing a model in which there are no κ-Aronszajn trees for any regular $\kappa > \aleph_1$, we produce a model with many singular cardinals where both the singular cardinals hypothesis and weak square fail.

1. Introduction

In this paper, we produce a model of ZFC with some global behavior of the continuum function on singular cardinals and the failure of weak square. Our method is as an extension of Sinapova's work [17]. We define a diagonal supercompact Radin forcing which adds a club subset to a cardinal κ while forcing the failure of SCH everywhere on the club and preserving the inaccessibility of κ. In the forcing extension, weak square will necessarily hold at some successors of singular cardinals below κ, but the set of these singular cardinals will be sufficiently sparse that it can be made non-stationary by κ-distributive forcing. We will thus obtain the following result.

Theorem 1.1. *If there are a supercompact cardinal κ and a weakly inaccessible cardinal $\theta > \kappa$, then there is a forcing extension in which κ is inaccessible and there is a club $E \subseteq \kappa$ of singular cardinals ν at which SCH and $\square^*_\nu$ both fail.*

We are motivated by the question of whether in ZFC one can construct a κ-Aronszajn tree for some $\kappa > \omega_1$. The question is also open if we ask for a *special* κ-Aronszajn tree. Forcing provides a possible path to a negative solution by showing that it is consistent with ZFC that there are no κ-Aronsajn trees on any regular $\kappa > \omega_1$. By a theorem of Jensen [11], $\square^*_\mu$ is equivalent to the existence of a special μ^+-Aronszajn tree. So our theorem is partial progress towards a model with no special Aronszajn trees.

The non-existence of κ-Aronszajn trees (the tree property at κ) and the non-existence of special κ-Aronszajn trees (failure of $\square^*$) are reflection principles which are closely connected with large cardinals. For example, theorems of Erdös and Tarski [6], and Monk and Scott [14], show that an inaccessible cardinal is weakly compact if and only if it has the tree property. Further, Mitchell and Silver [13] showed that the tree property at $\aleph_2$ is consistent with ZFC if and only if the existence of a weakly compact cardinal is.

Specker [21] showed that, if $\kappa^{<\kappa} = \kappa$, then there is a special κ^+-Aronszajn tree. This theorem places an important restriction on models where there are no special

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Aronszajn trees. From Specker's theorem, a model with no special κ-Aronszajn trees for any $\kappa > \aleph_1$ must be one in which GCH fails everywhere. In particular GCH must fail at every singular strong limit cardinal, a failure of the Singular Cardinals Hypothesis (SCH). The consistency of the failure of SCH requires large cardinals [8]; a model in which GCH fails everywhere was first obtained by Foreman and Woodin [7].

There are many partial results towards constructing a model in which every regular cardinal greater than $\aleph_1$ has the tree property. There is a bottom up approach where one attempts to force longer and longer initial segments of the regular cardinals to have the tree property; see, for example [1, 3, 16, 24]. We refer the reader to [22] for some analogous results on successive failures of weak square. Another aspect of the problem comes from the interaction between cardinal arithmetic at singular strong limit cardinals μ and the tree property at μ^+. In the 1980's Woodin asked whether the failure of SCH at $\aleph_\omega$ is consistent with the tree property at $\aleph_{\omega+1}$. More generally, one can consider whether this situation is consistent at some larger singular cardinal. An important result in this direction is due to Gitik and Sharon [10], who showed that, relative to the existence of a supercompact cardinal, it is consistent that there is a singular cardinal κ of cofinality ω such that SCH fails at κ and there are no special κ^+-Aronszajn trees. In fact they show a stronger assertion ($\kappa^+ \notin I[\kappa^+]$), which we will define later. In the same paper, they show that it is possible to make κ into $\aleph_{\omega^2}$. Cummings and Foreman [4] showed that there is a PCF theoretic object called a bad scale in the models of Gitik and Sharon, which implies that $\kappa^+ \notin I[\kappa^+]$.

The key ingredient in Gitik and Sharon's argument was a new diagonal supercompact Prikry forcing. The basic idea is to start with supercompactness measures U_n on $\mathcal{P}_\kappa(\kappa^{+n})$ for $n < \omega$ and use them to define a Prikry forcing. This forcing adds a sequence $\langle x_n \mid n < \omega\rangle$, where each x_n is a typical point for U_n and $\bigcup_{n<\omega} x_n = \kappa^{+\omega}$. The result is that $\kappa^{+\omega}$ is collapsed to have to have size κ and $\kappa^{+\omega+1}$ becomes the new successor of κ. The fact that $\kappa^{+\omega+1} \notin I[\kappa^{+\omega+1}]$ in the ground model persists to provide $\kappa^+ \notin I[\kappa^+]$ in the extension. Moreover, if we start with $2^\kappa = \kappa^{+\omega+2}$ in the ground model, then we get the failure of SCH at κ in the extension.

Variations of Gitik and Sharon's poset have been used to construct many related models. We list a few such results:

(1) (Neeman [15]) From ω supercompact cardinals, there is a forcing extension in which there is a singular cardinal κ of cofinality ω such that SCH fails at κ and κ^+ has the tree property.
(2) (Sinapova [17]) From a supercompact cardinal κ, for any regular $\lambda < \kappa$, there is a forcing extension in which κ is a singular cardinal of cofinality λ, SCH fails at κ and κ carries a bad scale (in particular $\kappa^+ \notin I[\kappa^+]$ and there are no special κ^+-Aronszajn trees).
(3) (Sinapova [18]) From λ supercompact cardinals $\langle \kappa_\alpha \mid \alpha < \lambda\rangle$ with $\lambda < \kappa_0$ regular, there is a forcing extension in which κ_0 is a singular cardinal of cofinality λ, SCH fails at κ_0 and κ_0^+ has the tree property.
(4) (Sinapova [19]) From ω supercompact cardinals, it is consistent that Neeman's result above holds with $\kappa = \aleph_{\omega^2}$.

Woodin's original question remains open; see [20] for the best known partial result. A theme in the above results is that questions about the tree property are answered by first constructing a model where there are no special κ^+-Aronszajn trees (or even $\kappa^+ \notin I[\kappa^+]$). To obtain the tree property, one needs to increase the large cardinal assumption and to give a version of an argument of Magidor and Shelah [12], who showed that the tree property holds at μ^+ when μ is a singular limit of supercompact cardinals. The results of our paper are based on the ideas from Sinapova's [17], but we expect that they will generalize to give the tree property in the presence of stronger large cardinal assumptions.

The paper is organized as follows. In Section 2 we give some definitions and background material required for the main result. In Section 3 we describe the main forcing for Theorem 1.1 and prove that it gives a model with a club C of cardinals where SCH fails. In Section 4 we characterize which cardinals in the club C have weak square sequences and show that this set can be made non-stationary by κ-distributive forcing, thus completing the proof of Theorem 1.1. In Section 5 we make some concluding remarks and ask some open questions.

2. Background

In this section we will make the notions from the introduction precise and give some further definitions that are relevant to the rest of the paper.

Definition We say that ν *has a weak square sequence* ($\square^*_\nu$) if there is a sequence $\langle C_\gamma \mid \gamma < \nu^+\rangle$ such that

1. for all $\gamma < \nu^+$ limit, $C_\gamma \subset \mathcal{P}(\gamma)$ is nonempty of size at most ν such that, for every $c \in C_\gamma$, $c \subset \gamma$ is club in γ with $\mathrm{otp}(c) \leq \nu$, and
2. for all $\beta < \gamma < \nu$, if β is a limit point of some $c \in C_\gamma$, then $c \cap \beta \in C_\beta$.

Definition Let $\vec{z} = \langle z_\alpha \mid \alpha < \nu^+\rangle$ be a sequence of bounded subsets of ν^+. We say that a limit ordinal γ is $\vec{z}$-approachable if there is an unbounded set $A \subset \gamma$ with $\mathrm{otp}(A) = \mathrm{cf}(\gamma)$ such that, for every $\beta < \gamma$, $A \cap \beta = z_\alpha$ for some $\alpha < \gamma$. The approachability ideal $I[\nu^+]$ consists of all subsets $S \subset \nu^+$ for which there are $\vec{z}$ as above and a club $C \subset \nu^+$ so that every $\gamma \in C \cap S$ is $\vec{z}$-approachable.

By arranging that $z_{\alpha+1}$ is the closure of z_α for each $\alpha < \nu^+$, we may assume that for every $\vec{z}$-approachable point γ, there is a witness $A \subset \gamma$ which is closed.

2.1. Forcing preliminaries.

Suppose κ is supercompact and $\theta > \kappa$. As in [17], we can arrange that $2^\kappa \geq \theta$ and that we have a sequence of ultrafilters $\vec{U} = \langle U_\alpha \mid \alpha < \theta\rangle$ and, for all $\alpha < \theta$, a sequence $\langle f^\alpha_\eta \mid \eta < \theta\rangle$ such that the following hold.

- For all $\alpha < \theta$, U_α is a normal, fine ultrafilter on $\mathcal{P}_\kappa(\kappa^{+\alpha})$. Let $j_\alpha : V \to M_\alpha \cong \mathrm{Ult}(V, U_\alpha)$ be the collapsed ultrapower map.
- For all $\alpha < \beta < \theta$, $U_\alpha \in M_\beta$.
- For all $\alpha, \eta < \theta$, we have $f^\alpha_\eta : \kappa \to \kappa$ and $j_\alpha(f^\alpha_\eta)(\kappa) = \eta$.

When we write that something happens for most (or for almost all) $x \in \mathcal{P}_\kappa(\kappa^{+\alpha})$, we mean it happens for a U_α-measure one set. For $\alpha < \theta$, for most $x \in \mathcal{P}_\kappa(\kappa^{+\alpha})$, $x \cap \kappa$ is an inaccessible cardinal. We will always work with such x and will write κ_x for $x \cap \kappa$. For $x, y \in \mathcal{P}_\kappa(\kappa^{+\alpha})$, $x \prec y$ denotes the statement that $x \subseteq y$ and $\mathrm{otp}(x) < \kappa_y$.

For $\alpha < \beta < \theta$, let $\bar{u}^\beta_\alpha$ be a function on $\mathcal{P}_\kappa(\kappa^{+\beta})$ representing U_α in the ultrapower by U_β. For most $x \in \mathcal{P}_\kappa(\kappa^{+\beta})$, $\bar{u}^\beta_\alpha(x)$ is a measure on $\mathcal{P}_{\kappa_x}(\kappa_x^{+f^\beta_\alpha(\kappa_x)})$. Also, for most $x \in \mathcal{P}_\kappa(\kappa^{+\beta})$, $\mathrm{otp}(x \cap \kappa^{+\alpha}) = f^\beta_\alpha(\kappa_x)$. For such x, $\bar{u}^\beta_\alpha(x)$ lifts naturally to a measure $u^\beta_\alpha(x)$ on $\mathcal{P}_{\kappa_x}(x \cap \kappa^{+\alpha})$.

For $y \in \mathcal{P}_\kappa(\kappa^{+\beta})$, let $Z^\beta_y = \{\alpha < \beta \mid \kappa^{+\alpha} \in y\}$.

Lemma 2.1. *For most $y \in \mathcal{P}_\kappa(\kappa^{+\beta})$, the following hold.*

(1) *Z^β_y is $< \kappa_y$-closed.*
(2) *If $\mathrm{cf}(\beta) < \kappa$, then $\mathrm{cf}(\beta) < \kappa_y$ and Z^β_y is unbounded in β.*
(3) *$\mathrm{otp}(Z^\beta_y) = f^\beta_\beta(\kappa_y)$ and, if β is a limit ordinal, then so is $f^\beta_\beta(\kappa_y)$. Also, if $cf(\beta) \geq \kappa$ then $\mathrm{cf}(f^\beta_\beta(\kappa_y)) \geq \kappa_y$.*
(4) *For all $\alpha \in Z^\beta_y$, $\mathrm{otp}(y \cap \kappa^{+\alpha}) = \kappa_y^{+f^\beta_\alpha(\kappa_y)} = \kappa_y^{+\mathrm{otp}(\alpha \cap Z^\beta_y)}$.*
(5) *For all $\alpha \in Z^\beta_y$, $\bar{u}^\beta_\alpha(y)$ is a measure on $\mathcal{P}_{\kappa_y}(\kappa_y^{+f^\beta_\alpha(\kappa_y)})$.*
(6) *For all $\alpha_0 < \alpha_1$, both in Z^β_y, the function $x \mapsto \bar{u}^{\alpha_1}_{\alpha_0}(x)$ represents $\bar{u}^\beta_{\alpha_0}(y)$ in the ultrapower by $u^\beta_{\alpha_1}(y)$.*

Proof. Let $j = j_\beta$. Recall that a set A is in U_β iff $j``\kappa^{+\beta} \in j(A)$. Note first that, defining $g : \mathcal{P}_\kappa(\kappa^{+\beta}) \to V$ by $g(y) = Z^\beta_y$, we have $j(g)(j``\kappa^{+\beta}) = j``\beta$. Items (1)-(4) then follow easily.

To show (5), let $j(\langle \bar{u}^\beta_\alpha \mid \alpha < \beta \rangle) = \langle \bar{v}^{j(\beta)}_\alpha \mid \alpha < j(\beta) \rangle$ and $j(\langle f^\beta_\alpha \mid \alpha < \beta \rangle) = \langle g^{j(\beta)}_\alpha \mid \alpha < j(\beta) \rangle$. It suffices to show that, in M_β, for all $\alpha \in j``\beta$, $\bar{v}^{j(\beta)}_\alpha(j``\kappa^{+\beta})$ is a measure on $\mathcal{P}_\kappa(\kappa^{+g^{j(\beta)}_\alpha(\kappa)})$. Let $\alpha \in j``\beta$, with, say, $\alpha = j(\xi)$. Then $\bar{v}^{j(\beta)}_\alpha(j``\kappa^{+\beta}) = j(\bar{u}^\beta_\xi)(j``\kappa^{+\beta}) = U_\xi$, which is a measure on $\mathcal{P}_\kappa(\kappa^{+\xi}) = \mathcal{P}_\kappa(\kappa^{+g^{j(\beta)}_\alpha(\kappa)})$.

We finally show (6). Let $j(\langle \bar{u}^{\alpha_1}_{\alpha_0} \mid \alpha_0 < \alpha_1 \leq \beta \rangle) = \langle \bar{v}^{\alpha_1}_{\alpha_0} \mid \alpha_0 < \alpha_1 \leq j(\beta) \rangle$ and $j(\langle u^\beta_\alpha \mid \alpha < \beta \rangle) = \langle v^{j(\beta)}_\alpha \mid \alpha < j(\beta) \rangle$. It suffices to show that, in M_β, for all $\alpha_0 < \alpha_1$, both in $j``\beta$, the function $x \mapsto \bar{v}^{\alpha_1}_{\alpha_0}(x)$ represents $\bar{v}^{j(\beta)}_{\alpha_0}(j``\kappa^{+\beta})$ in the ultrapower by $v^{j(\beta)}_{\alpha_1}(j``\kappa^{+\beta})$. Fix $\alpha_0 < \alpha_1$ in $j``\beta$, with $\alpha_0 = j(\xi_0)$ and $\alpha_1 = j(\xi_1)$. Note that $v^{j(\beta)}_{\alpha_1}(j``\kappa^{+\beta})$ is a measure on $\mathcal{P}_\kappa(j``\kappa^{+\xi_1})$ that collapses to U_{ξ_1}. Call this 'lifted' measure $\hat{U}_{\xi_1}$. Also note that $\bar{v}^{j(\beta)}_{\alpha_0}(j``\kappa^{+\beta}) = U_{\xi_0}$. Thus, we must show that the function $x \mapsto \bar{v}^{\alpha_1}_{\alpha_0}(x)$ represents U_{ξ_0} in the ultrapower by $\hat{U}_{\xi_1}$.

Fix $x \in \mathcal{P}_\kappa(j``\kappa^{+\xi_1})$. There is $\bar{x} \in \mathcal{P}_\kappa(\kappa^{+\beta})$ such that $x = j(\bar{x})$. Then $\bar{v}^{\alpha_1}_{\alpha_0}(x) = j(\bar{u}^{\xi_1}_{\xi_0}(\bar{x}))$. For most $\bar{x} \in \mathcal{P}_\kappa(\kappa^{+\xi_1})$, $\bar{u}^{\xi_1}_{\xi_0}(\bar{x})$ is a measure on $\mathcal{P}_{\kappa_{\bar{x}}}(\kappa_{\bar{x}}^{+f^{\xi_1}_{\xi_0}(\kappa_{\bar{x}})})$, and this is fixed by j. Thus, for most $x \in \mathcal{P}_\kappa(j``\kappa^{+\xi_1})$, $\bar{v}^{\alpha_1}_{\alpha_0}(x) = \bar{u}^{\xi_1}_{\xi_0}(\bar{x})$. Therefore, since $\hat{U}_{\xi_1}$ is a lifting of U_{ξ_1}, $x \mapsto \bar{v}^{\alpha_1}_{\alpha_0}(x)$ represents the same thing in the ultrapower by $\hat{U}_{\xi_1}$ as $\bar{x} \mapsto \bar{u}^{\xi_1}_{\xi_0}(x)$ represents in the ultrapower by U_{ξ_1}, which is U_{ξ_0}. This is true in V and, since M_β is sufficiently closed, it is true in M_β as well. □

Lemma 2.2. *Suppose $\beta < \theta$ and, for all $\alpha \leq \beta$, $A_\alpha \in U_\alpha$. Let A^* be the set of all $y \in A_\beta$ such that, for all $\alpha \in Z^\beta_y$, $\{x \in A_\alpha \mid x \prec y\} \in u^\beta_\alpha(y)$. Then $A^* \in U_\beta$.*

Proof. Let $j = j_\beta$. It suffices to show that $j``\kappa^{+\beta} \in j(A^*)$, i.e. for all $j(\alpha) \in j``\beta$, $\{x \in j(A_\alpha) \mid x \prec j``\kappa^{+\beta}\} \in \hat{U}_\alpha$, where $\hat{U}_\alpha$ is the lifted version of $U_\alpha$ living on $\mathcal{P}_\kappa(j``\kappa^{+\alpha})$. Fix such a $j(\alpha)$. Let $X = \{x \in j(A_\alpha) \mid x \prec j``\kappa^{+\beta}\}$, and note that $X = j``A_\alpha \in \hat{U}_\alpha$. □

Lemma 2.3. *Suppose $\gamma < \theta$, $z \in \mathcal{P}_\kappa(\kappa^{+\gamma})$, and z satisfies all of the statements in Lemma 2.1. Suppose that, for all $\alpha \in Z^\gamma_z$, $A_\alpha \in u^\gamma_\alpha(z)$. Fix $\beta \in Z^\gamma_z$, and let A^* be the set of $y \in A_\beta$ such that, for all $\alpha \in Z^\beta_y$, $\{x \in A_\alpha \mid x \prec y\} \in u^\beta_\alpha(y)$. Then $A^* \in u^\gamma_\beta(z)$.*

Proof. For each $\alpha \in Z^\gamma_z$, let $\bar{A}_\alpha$ be the collapsed version of A_α, so $\bar{A}_\alpha \in \bar{u}^\gamma_\alpha(z)$. Recall that $u^\gamma_\beta(z)$ is a measure on $\mathcal{P}_{\kappa_z}(z \cap \kappa^{+\beta})$. Let $k : V \to N \cong \mathrm{Ult}(V, u^\gamma_\beta(z))$ be the ultrapower map. By (6) of Lemma 2.1, for all $\alpha \in Z^\gamma_z \cap \beta$, the map $y \mapsto \bar{u}^\beta_\alpha(y)$ represents $\bar{u}^\gamma_\alpha(z)$ in the ultrapower. Note also that the map $y \mapsto Z^\beta_y$ represents $\{\eta < k(\beta) \mid k(\kappa)^{+\eta} \in k``(z \cap \kappa^{+\beta})\} = k``(Z^\gamma_z \cap \beta)$. To prove the lemma, it suffices to show that $k``(z \cap \kappa^{+\beta}) \in k(A^*)$, i.e. for all $\alpha \in Z^\gamma_z \cap \beta$, $\{x \in k(A_\alpha) \mid x \prec k``(z \cap \kappa^{+\beta})\}$ is in the measure represented by the map $y \mapsto u^\beta_\alpha(y)$. Call this measure w and note that it is a lifted version of $\bar{u}^\gamma_\alpha(z)$. Also note that $\{x \in k(A_\alpha) \mid x \prec k``(z \cap \kappa^{+\beta})\} = k``(A_\alpha)$, which collapses to $\bar{A}_\alpha \in \bar{u}^\gamma_\alpha(z)$. Thus, $\{x \in k(A_\alpha) \mid x \prec k``(z \cap \kappa^{+\beta})\} \in w$, completing the proof of the lemma. □

3. The main forcing

For $\beta < \theta$, let X_β be the set of $y \in \mathcal{P}_\kappa(\kappa^{+\beta})$ satisfying all of the statements in Lemma 2.1. Fix $\eta < \theta$. We define a forcing notion, $\mathbb{P}_{\vec{U},\eta}$. Conditions of $\mathbb{P}_{\vec{U},\eta}$ are pairs (a, A) satisfying the following requirements.

(1) a and A are functions, $\mathrm{dom}(a)$ is a finite subset of $\theta \setminus \eta$, and $\mathrm{dom}(A) = \theta \setminus (\mathrm{dom}(a) \cup \eta)$.
(2) For all $\beta \in \mathrm{dom}(a)$, $a(\beta) \in X_\beta$.
(3) For all $\alpha < \beta$, both in $\mathrm{dom}(a)$, $a(\alpha) \prec a(\beta)$ and $\alpha \in Z^\beta_{a(\beta)}$.
(4) For all $\alpha \in \theta \setminus (\max(\mathrm{dom}(a)) + 1)$ (or, if $\mathrm{dom}(a) = \emptyset$, for all $\alpha \in \mathrm{dom}(A)$), $A(\alpha) \in U_\alpha$.
(5) For all $\alpha \in \mathrm{dom}(A) \cap \max(\mathrm{dom}(a))$, if $\beta = \min(\mathrm{dom}(a) \setminus \alpha)$, then $A(\alpha) \in u^\beta_\alpha(a(\beta))$ if $\alpha \in Z^\beta_{a(\beta)}$ and $A(\alpha) = \emptyset$ if $\alpha \notin Z^\beta_{a(\beta)}$.
(6) For all $\beta \in \mathrm{dom}(A)$ such that $A(\beta) \neq \emptyset$ and $\mathrm{dom}(a) \cap \beta \neq \emptyset$, if $\alpha = \max(\mathrm{dom}(a) \cap \beta)$, then for all $y \in A(\beta)$, $a(\alpha) \prec y$ and $\alpha \in Z^\beta_y$.

If $(a, A), (b, B) \in \mathbb{P}_{\vec{U},\eta}$, then $(b, B) \leq (a, A)$ iff the following requirements hold.

(1) $b \supseteq a$.
(2) For all $\alpha \in \mathrm{dom}(b) \setminus \mathrm{dom}(a)$, $b(\alpha) \in A(\alpha)$.
(3) For all $\alpha \in \mathrm{dom}(B)$, $B(\alpha) \subseteq A(\alpha)$.

$(b, B) \leq^* (a, A)$ if $(b, B) \leq (a, A)$ and $b = a$. In this case, (b, B) is called a *direct extension* of (a, A).

Remark In our arguments, for notational simplicity we will typically assume $\eta = 0$ and then denote $\mathbb{P}_{\vec{U},\eta}$ as $\mathbb{P}_{\vec{U}}$. Everything proved about $\mathbb{P}_{\vec{U}}$ can be proved for a general $\mathbb{P}_{\vec{U},\eta}$ in the same way by making the obvious changes. The reason we introduce the more general forcing is to be able to properly state the Factorization Lemma.

In what follows, let $\mathbb{P}$ denote $\mathbb{P}_{\vec{U}}$. For any condition $p = (a, A) \in \mathbb{P}$, we often denote (a, A) as (a^p, A^p) and let $\gamma^p = \max(\mathrm{dom}(a^p))$. We refer to a^p as the *stem* of p. Note that, if $p, q \in \mathbb{P}$ and $a^p = a^q$, then p and q are compatible. If a is a non-empty stem, then let γ^a denote $\max(\mathrm{dom}(a))$, and let $a^- = a \restriction \gamma^a$. Suppose a is a stem, $\alpha < \theta$, and $x \in X_\alpha$. Suppose moreover that either a is empty or $\gamma^a < \alpha$, $a(\gamma^a) \prec x$, and $\gamma^a \in Z^\alpha_x$. Then $a^\frown(\alpha, x)$ is a stem and $(a^\frown(\alpha, x))^- = a$. If $p \in \mathbb{P}$ and b is a stem, then b is *possible* for p if there is $q \leq p$ with $a^q = b$. If $p \in \mathbb{P}$ and b is possible for p, then $b^\frown p$ is the maximal q such that $q \leq p$ and $a^q = b$. Such a q always exists.

Lemma 3.1. *Suppose $(a, A) \in \mathbb{P}$, $\beta \in \mathrm{dom}(A)$, and $A(\beta) \neq \emptyset$. Then there is $(b, B) \leq (a, A)$ such that $\beta \in \mathrm{dom}(b)$.*

Proof. Straightforward using Lemmas 2.1, 2.2, and 2.3. □

Definition Suppose that G is a $\mathbb{P}$-generic over V. Let C^{sc}_G (sc for supercompact) be the set of all points $x = a(\beta)$ where $\beta \in \mathrm{dom}(a)$ for some $p = (a, A)$ in the generic filter G, and let $C_G = \{\kappa_x \mid x \in C^{\mathrm{sc}}_G\}$ be the generic Radin club.

Lemma 3.2. *C_G is club in κ and the assignment $x \mapsto \kappa_x = x \cap \kappa$ is an increasing bijection from C^{sc}_G and C_G.*

Proof. Straightforward by Lemma 3.1 and genericity. □

Suppose $\beta < \theta$ and $y \in X_\beta$. Let $\vec{U}_y = \langle \bar{u}^\beta_\alpha(y) \mid \alpha \in Z^\beta_y \rangle$. For $\xi < f^\beta_\beta(\kappa_y)$, let $\alpha_\xi \in Z^\beta_y$ be such that $\mathrm{otp}(\alpha \cap Z^\beta_y) = \xi$. Then $\bar{u}^\beta_{\alpha_\xi}(y)$ is a measure on $\mathcal{P}_{\kappa_y}(\kappa_y^{+\xi})$. Let $V_\xi = \bar{u}^\beta_{\alpha_\xi}(y)$. Then $\vec{U}_y = \langle V_\xi \mid \xi < f^\beta_\beta(y) \rangle$, and we can define $\mathbb{P}_{\vec{U}_y}$ as above.

If $p \in \mathbb{P}$, then $\mathbb{P}/p = \{q \in \mathbb{P} \mid q \leq p\}$.

Lemma 3.3. *(Factorization Lemma) Let $p = (a, A) \in \mathbb{P}$. Suppose $a \neq \emptyset$, $\gamma = \gamma^a$, and $y = a(\gamma)$. Then there is $p' \in \mathbb{P}_{\vec{U}_y}$ such that $\mathbb{P}/p \cong \mathbb{P}_{\vec{U}_y}/p' \times \mathbb{P}_{\vec{U},\gamma+1}/(\emptyset, A \restriction (\gamma, \theta))$.*

Proof. Let $\pi : y \to \mathrm{otp}(y)$ be the unique order-preserving bijection. Define $p' = (a', A') \in \mathbb{P}_{\vec{U}_y}$ as follows. For $\xi < f^\gamma_\gamma(y)$, let $\alpha_\xi \in Z^\gamma_y$ be such that $\mathrm{otp}(\alpha_\xi \cap Z^\gamma_y) = \xi$. Let $\mathrm{dom}(a') = \{\xi < f^\gamma_\gamma(y) \mid \alpha_\xi \in \mathrm{dom}(a)\}$ and, for $\xi \in \mathrm{dom}(a')$, let $a'(\xi) = \pi``a(\alpha_\xi)$. Then $\mathrm{dom}(A') = f^\gamma_\gamma(y) \setminus \mathrm{dom}(a')$. If $\xi \in \mathrm{dom}(A')$, let $A'(\xi) = \{\pi``x \mid x \in A(\alpha_\xi)\}$. It is straightforward to verify that p' thus defined is in $\mathbb{P}_{\vec{U}_y}$ and that $\mathbb{P}/p \cong \mathbb{P}_{\vec{U}_y}/p' \times \mathbb{P}_{\vec{U},\gamma+1}/(\emptyset, A \restriction (\gamma, \theta))$. □

By repeatedly applying the Factorization Lemma, standard arguments (see, e.g. [9]) allow us to assume we are working below a condition of the form $(\emptyset, A)$ when proving the following lemmas about $\mathbb{P}$.

Lemma 3.4. *$(\mathbb{P}, \leq, \leq^*)$ satisfies the Prikry property, i.e. if φ is a statement in the forcing language and $p \in \mathbb{P}$, then there is $q \leq^* p$ such that $q \parallel \varphi$.*

Proof. The proofs of this and the next few lemmas are similar to those for the classical Radin forcing, which can be found in [9]. Fix φ in the forcing language and $p \in \mathbb{P}$. By the Factorization Lemma, we may assume that $p = (\emptyset, A)$ for some A. Let a be a stem possible for p, and let $\alpha \in \theta \setminus (\gamma^a + 1)$. Let $Y_{a,\alpha} = \{x \in A(\alpha) \mid a(\gamma^a) \prec x$ and $\gamma^a \in Z^\alpha_x\}$. Note that $Y_{a,\alpha} \in U_\alpha$. Let $Y^0_{a,\alpha} = \{x \in Y_{a,\alpha} \mid$ for some B, $(a^\frown(\alpha, x), B) \Vdash \varphi\}$, $Y^1_{a,\alpha} = \{x \in Y_{a,\alpha} \mid$ for some B, $(a^\frown(\alpha, x), B) \Vdash \neg\varphi\}$, and $Y^2_{a,\alpha} = Y_{a,\alpha} \setminus (Y^0_{a,\alpha} \cup Y^1_{a,\alpha})$. Fix $i(a, \alpha) < 3$ such that $Y^{i(a,\alpha)}_{a,\alpha} \in U_\alpha$, and let $Y^*_{a,\alpha} = Y^{i(a,\alpha)}_{a,\alpha}$.

For $\alpha < \theta$, let $B(\alpha)$ be the set of $x \in A(\alpha)$ such that, for every stem a possible for p such that $a(\gamma^a) \prec x$ and $\gamma^a \in Z^\alpha_x$, $x \in Y^*_{a,\alpha}$. We claim that $B(\alpha) \in U_\alpha$. Let $j = j_\alpha$. It suffices to show that $j``\kappa^{+\alpha} \in j(B(\alpha))$. Let $j(\langle Y^*_{a,\alpha} \mid a$ is a stem possible for p and $\gamma^a < \alpha \rangle) = \langle W^*_{a,j(\alpha)} \mid a$ is a stem in $j(\mathbb{P})$ possible for $j(p)$ and $\gamma^a < j(\alpha)\rangle$. Suppose that, in $j(\mathbb{P})$, a is a stem possible for $j(p)$ such that $a(\gamma^a) \prec j``\kappa^{+\alpha}$ and $\gamma^a \in j``\alpha$. Then there is a stem $\bar{a}$ possible for p such that

$j(\bar{a}) = a$. Then $W^*_{a,j(\alpha)} = j(Y^*_{\bar{a},\alpha})$, so, as $Y^*_{\bar{a},\alpha} \in U_\alpha$, $j``\kappa^{+\alpha} \in W^*_{a,j(\alpha)}$, and hence $j``\kappa^{+\alpha} \in j(B(\alpha))$. Thus, $(\emptyset, B) \in \mathbb{P}$ and $(\emptyset, B) \leq^* p$.

Suppose for sake of contradiction that no direct extension of $(\emptyset, B)$ decides φ. Find $(a, B^*) \leq (\emptyset, B)$ deciding φ with $|a|$ minimal. Without loss of generality, suppose $(a, B^*) \Vdash \varphi$. Because of our assumption that no direct extension of $(\emptyset, B)$ decides φ, a is non-empty. Let $b = a^-$ and $\gamma = \gamma^a$. By our construction of B, we have $a(\gamma) \in Y^0_{b,\gamma}$, and, for any $x \in B(\gamma)$ such that $b^\frown(\gamma, x)$ is a stem, there is $\hat{B}_x$ such that $(b^\frown(\gamma, x), \hat{B}_x) \Vdash \varphi$. Let $p^* = b^\frown(\emptyset, B) = (b, B^*)$. We will find a direct extension (b, F) of p^* forcing φ, thus contradicting the minimality of $|a|$.

We first define $F \restriction \gamma^b$ (if $b = \emptyset$, then there is nothing to do here). Since there are fewer than κ possibilities for $F \restriction \gamma^b$ and U_γ is κ-complete, we may fix a function F^* on $\gamma^b \setminus \mathrm{dom}(b)$ such that $B_0(\gamma) := \{x \in B(\gamma) \mid b^\frown(\gamma, x) \text{ is a stem and } \hat{B}_x \restriction \gamma^b = F^*\} \in U_\gamma$. Then, for all $\alpha \in \gamma^b \setminus \mathrm{dom}(b)$, let $F(\alpha) = F^*(\alpha) \cap B^*(\alpha)$. We next define F on the interval (γ^b, γ) (or on all of γ, if $b = \emptyset$). If $\alpha \in (\gamma^b, \gamma)$, $x \in B_0(\gamma)$, and $\alpha \in Z^\gamma_x$, note that $\hat{B}_x(\alpha) \in u^\gamma_\alpha(x)$. Let $\bar{B}_x(\alpha)$ be the collapsed version of $\hat{B}_x(\alpha)$. Then $\bar{B}_x(\alpha) \in \bar{u}^\gamma_\alpha(x)$. Let $F^*(\alpha)$ be the set in U_α represented by the function $x \mapsto \bar{B}_x(\alpha)$ in the ultrapower by U_γ, and let $F(\alpha) = F^*(\alpha) \cap B^*(\alpha)$. Let $F(\gamma)$ be the set of $x \in B_0(\gamma) \cap B^*(\gamma)$ such that, for all $\alpha \in Z^\gamma_x \setminus (\gamma^b + 1)$, $\{y \in F(\alpha) \mid y \prec x\} = \hat{B}_x(\alpha)$. We claim that $F(\gamma) \in U_\gamma$. To see this, let $j = j_\gamma$. Note that the function $x \mapsto \hat{B}_x(\alpha)$ represents $\{j``y \mid y \in F(\alpha)\}$, which is equal to $\{z \in j(F(\alpha)) \mid z \prec j``\kappa^{+\gamma}\}$. Thus, $j``\kappa^{+\gamma} \in j(F(\gamma))$, so $F(\gamma) \in U_\gamma$. We finally define F on (γ, θ). If $\alpha \in (\gamma, \theta)$, let $F(\alpha)$ be the set of $y \in B^*(\alpha)$ such that $\gamma \in Z^\alpha_y$ and, for all $x \in F(\gamma)$ such that $x \prec y$, $y \in \hat{B}_x(\alpha)$. By now-familiar arguments, $F(\alpha) \in U_\alpha$. Notice that, by our construction, if $(c, H) \leq (b, F)$ and $\gamma \in \mathrm{dom}(c)$, then $(c, H) \leq (b^\frown(\gamma, c(\gamma)), \hat{B}_{c(\gamma)})$.

Now suppose for sake of contradiction that $(b, F) \nVdash \varphi$. Find $(c, H) \leq (b, F)$ such that $(c, H) \Vdash \neg\varphi$. If $\gamma \in \mathrm{dom}(c)$, then $(c, H) \leq (b^\frown(\gamma, c(\gamma)), \hat{B}_{c(\gamma)}) \Vdash \varphi$, which is a contradiction. Thus, suppose $\gamma \notin \mathrm{dom}(c)$. By our choice of $F(\alpha)$ for $\alpha \in (\gamma, \theta)$ (namely, our requirement that $\gamma \in Z^\alpha_y$ for all $y \in F(\alpha)$), it must be the case that $H(\gamma) \neq \emptyset$. But then (c, H) can be extended further to a condition (c', H') such that $\gamma \in \mathrm{dom}(c')$, and this again gives a contradiction. □

Definition A tree $T \subseteq [\bigcup_{\alpha<\theta}(\{\alpha\} \times \mathcal{P}_\kappa(\kappa^{+\alpha}))]^{\leq n}$ is *fat* if the following conditions hold.

(1) For all $\langle(\alpha_i, x_i) \mid i \leq k\rangle \in T$ and all $i_0 < i_1 \leq k$, we have $\alpha_{i_0} < \alpha_{i_1}$ and $x_{i_0} \prec x_{i_1}$.
(2) For all $t \in T$ with $\mathrm{lh}(t) < n$, there is $\alpha_t < \theta$ such that:
   (a) for all (β, y) such that $t^\frown(\beta, y) \in T$, $\beta = \alpha_t$;
   (b) $\{x \mid t^\frown(\alpha_t, x) \in T\} \in U_{\alpha_t}$.

If T is as in the previous definition, then n is said to be the *height* of T.

Definition Suppose T is a fat tree, $\alpha < \theta$, and $x \in X_\alpha$. T is *fat above* (α, x) *if, for all* $\langle(\alpha_i, x_i) \mid i \leq k\rangle \in T$ *and all* $i \leq k$, *we have* $\alpha < \alpha_i$ *and* $x \prec x_i$.

Definition Suppose $\gamma < \theta$ and $z \in X_\gamma$. A tree $T \subseteq [\bigcup_{\alpha<\theta}(\{\alpha\} \times \mathcal{P}_\kappa(\kappa^{+\alpha}))]^{\leq n}$ is *fat below* (γ, z) if the following conditions hold.

(1) For all $\langle(\alpha_i, x_i) \mid i \leq k\rangle \in T$ and all $i \leq k$, we have $\alpha_i \in Z^\gamma_z$ and $x_i \prec z$.
(2) For all $\langle(\alpha_i, x_i) \mid i \leq k\rangle \in T$ and all $i_0 < i_1 \leq k$, we have $\alpha_{i_0} < \alpha_{i_1}$ and $x_{i_0} \prec x_{i_1}$.

(3) For all $t \in T$ with $\mathrm{lh}(t) < n$, there is $\alpha_t \in Z^\gamma_z$ such that:
   (a) for all (β, y) such that $t^\frown(\beta, y) \in T$, $\beta = \alpha_t$;
   (b) $\{x \mid t^\frown(\alpha_t, x) \in T\} \in u^\gamma_{\alpha_t}(z)$.

Suppose T is a fat tree, $\gamma < \theta$, and $z \in \mathcal{P}_\kappa(\kappa^{+\gamma})$. $T \restriction (\gamma, z)$ is the subtree of T consisting of all $\langle(\alpha_i, x_i) \mid i \leq k\rangle \in T$ such that, for all $i \leq k$, $\alpha_i \in Z^\gamma_z$ and $x_i \prec z$. Let $\gamma_T = \sup(\{\alpha \mid \text{for some } \langle(\alpha_i, x_i) \mid i \leq k\rangle \in T \text{ and } i \leq k,\ \alpha = \alpha_i\})$. Note that, if θ is weakly inaccessible and $|\mathcal{P}_\kappa(\kappa^{+\alpha})| < \theta$ for all $\alpha < \theta$, we have $\gamma_T < \theta$.

Suppose that S is a set of stems and, for all $a \in S$, T_a is a fat tree above $(\gamma^a, a(\gamma^a))$. For $\gamma < \theta$, let $S_{<\gamma} = \{a \in S \mid \gamma^a < \gamma\}$. Let $E = \{\gamma < \theta \mid \text{for all } a \in S_{<\gamma},\ \gamma_{T_a} < \gamma\}$. Note that, if θ is weakly inaccessible, E is club in θ. For all $\gamma \in E$, let Y_γ be the set of $z \in X_\gamma$ such that, for all $a \in S_{<\gamma}$ such that $a(\gamma^a) \prec z$ and $\gamma^a \in Z^\gamma_z$, $T_a \restriction (\gamma, z)$ is fat below (γ, z). We claim that $Y_\gamma \in U_\gamma$. To see this, let $j = j_\gamma$, and note first that $\{a \in j(S_{<\gamma}) \mid a \prec j``\kappa^{+\gamma} \text{ and } \gamma^a \in j``\gamma\} = j``S_{<\gamma}$ and second that, for all $\bar{a} \in S_{<\gamma}$, $j(T_{\bar{a}}) \restriction (j(\gamma), j``\kappa^{+\gamma}) = j``T_{\bar{a}}$, which is fat below $(j(\gamma), j``\kappa^{+\gamma})$. Thus, $j``\kappa^{+\gamma} \in j(Y_\gamma)$, so $Y_\gamma \in U_\gamma$.

Lemma 3.5. *Suppose $p = (a, A) \in \mathbb{P}$ and $D \subseteq \mathbb{P}$ is a dense open set. Suppose $a = \{(\alpha_i, x_i) \mid i < k\}$ is such that, for all $i_0 < i_1 < k$, $\alpha_{i_0} < \alpha_{i_1}$. There are trees $\langle T_i \mid i \leq k\rangle$ and natural numbers $\langle n_i \mid i \leq k\rangle$ such that the following hold.*

(1) *For all $i \leq k$, T_i is a tree of height n_i.*
(2) *If $k > 0$, then T_0 is fat below (α_0, x_0) and, for all $0 < i < k$, T_i is fat below (α_i, x_i) and above (α_{i-1}, x_{i-1}).*
(3) *T_k is fat, and, if $k > 0$, it is above (α_{k-1}, x_{k-1}).*
(4) *Suppose that, for all $i \leq k$, $\langle(\beta^i_\ell, y^i_\ell) \mid \ell < n_i\rangle$ is a maximal element of T_i. Then, if $b = a \cup \{(\beta^i_\ell, y^i_\ell) \mid i \leq k, \ell \leq n_i\}$, there is B such that $(b, B) \leq (a, A)$ and $(b, B) \in D$.*

Proof. By the Factorization Lemma, it again suffices to consider p of the form $(\emptyset, A)$. We thus need to find a single fat tree T. We inductively construct a decreasing sequence of conditions $\langle(\emptyset, A_n) \mid n < \omega\rangle$. Intuitively, A_n will take care of extensions $(b, B) \leq (\emptyset, A)$ such that $|b| = n$. We explicitly go through the first few steps of the construction.

Let $A_0 = A$. If there is a direct extension of $(\emptyset, A)$ in D, then we are done by setting $T = \{\emptyset\}$. Thus, suppose there is no such direct extension. For every stem a possible for $(\emptyset, A_0)$ and every $\alpha \in (\gamma^a, \theta)$, let $Y_{0,a,\alpha} = \{x \in A_0(\alpha) \mid a(\gamma^a) \prec x \text{ and } \gamma^a \in Z^\alpha_x\}$. Let $Y^0_{0,a,\alpha} = \{x \in Y_{0,a,\alpha} \mid \text{for some } B,\ (a^\frown(\alpha, x), B) \in D\}$, and let $Y^1_{0,a,\alpha} = Y_{0,a,\alpha} \setminus Y^0_{0,a,\alpha}$. Find $i(0, a, \alpha) < 2$ such that $Y^{i(0,a,\alpha)}_{0,a,\alpha} \in U_\alpha$, and let $Y^*_{0,a,\alpha} = Y^{i(0,a,\alpha)}_{0,a,\alpha}$. For $\alpha < \theta$, let $A_1(\alpha)$ be the set of $x \in A_0(\alpha)$ such that, for all stems a possible for $(\emptyset, A_0)$ such that $a(\gamma^a) \prec x$ and $\gamma^a \in Z^\alpha_x$, $x \in Y^*_{0,a,\alpha}$. As in the proof of Lemma 3.4, $A_1(\alpha) \in U_\alpha$ for all $\alpha < \theta$, so $(\emptyset, A_1) \leq^* (\emptyset, A_0)$. Note that $(\emptyset, A_1)$ satisfies the following property, which we denote $(*)_1$:

> Suppose $q = (a^\frown(\alpha, x), B) \leq (\emptyset, A_1)$ and $q \in D$. Then, for every $y \in A_1(\alpha)$ such that $a(\gamma^a) \prec y$ and $\gamma^a \in Z^\alpha_y$, there is B_y such that $(a^\frown(\alpha, y), B_y) \in D$.

Now suppose there is a stem $a = \{(\alpha, x)\}$ possible for $(\emptyset, A_1)$ and a B such that $(a, B) \in D$. We can then define a fat tree T of height 1 whose maximal elements are all $\langle(\alpha, x)\rangle$ such that $x \in A_1(\alpha)$. We are then done, as T easily satisfies the requirements of the lemma. Thus, suppose there is no such a and proceed to define $(\emptyset, A_2)$ as follows.

For every stem a possible for $(\emptyset, A_1)$ and every $\alpha \in (\gamma^a, \theta)$, let $Y_{1,a,\alpha} = \{x \in A_1(\alpha) \mid a(\gamma^a) \prec x$ and $\gamma^a \in Z^\alpha_x\}$. Let $Y^0_{1,a,\alpha}$ be the set of all $x \in Y_{1,a,\alpha}$ such that there are $\beta^\alpha_x \in (\alpha, \theta)$ and $W^\alpha_x \in U_{\beta^\alpha_x}$ such that, for all $y \in W^\alpha_x$:

- $x \prec y$ and $\alpha \in Z^{\beta^\alpha_x}_y$;
- there is B such that $(a^\frown(\alpha, x)^\frown(\beta^\alpha_x, y), B) \in D$.

Let $Y^1_{1,a,\alpha} = Y_{1,a,\alpha} \setminus Y^0_{1,a,\alpha}$. Find $i(1, a, \alpha) < 2$ such that $Y^{i(1,a,\alpha)}_{1,a,\alpha} \in U_\alpha$, and let $Y^*_{1,a,\alpha} = Y^{i(1,a,\alpha)}_{1,a,\alpha}$. For $\alpha < \theta$, let $A_2(\alpha)$ be the set of $x \in A_1(\alpha)$ such that, for all stems a possible for $(\emptyset, A_1)$ such that $a(\gamma^a) \prec x$ and $\gamma^a \in Z^\alpha_x$, $x \in Y^*_{1,a,\alpha}$. Then $(\emptyset, A_2) \leq^* (\emptyset, A_1)$, and $(\emptyset, A_2)$ satisfies the property $(*)_2$:

> Suppose $q = (a^\frown(\alpha, x)^\frown(\beta, y), B) \leq (\emptyset, A_2)$ and $q \in D$. Then, for every $x' \in A_2(\alpha)$ such that $a(\gamma^a) \prec x'$ and $\gamma^a \in Z^\alpha_{x'}$, there is $\beta^\alpha_{x'} \in (\alpha, \theta)$ and $W^\alpha_{x'} \in U_{\beta^\alpha_{x'}}$ such that, for all $y' \in W^\alpha_{x'}$, there is B' such that $(a^\frown(\alpha, x')^\frown(\beta^\alpha_{x'}, y'), B') \in D$.

Suppose there is a stem $a = \{(\alpha, x), (\beta, y)\}$ possible for $(\emptyset, A_2)$ with $\alpha < \beta$ and a B such that $(a, B) \in D$. Using $(*)_2$, we can define a tree T of height 2 whose maximal elements are all $\langle(\alpha, x'), (\beta^\alpha_{x'}, y')\rangle$ such that $x' \in A_2(\alpha)$ and $y' \in W^\alpha_{x'}$. We are then done, as T satisfies the requirements of the lemma. If there is no such stem a, then continue in the same manner.

In this way, we can construct A_n such that, if there is a stem a possible for $(\emptyset, A_n)$ with $|a| = n$ and a B such that $(a, B) \in D$, then there is a fat tree of height n as desired. For $\alpha < \theta$, let $A_\infty(\alpha) = \bigcap_{n<\omega} A_n(\alpha)$. For all $n < \omega$, $(\emptyset, A_\infty) \leq^* (\emptyset, A_n)$. Find $(a, B) \leq (\emptyset, A_\infty)$ such that $(a, B) \in D$. Let $n^* = |a|$. Then a is possible for $(\emptyset, A_{n^*})$, so there is a fat tree of height n^* as required by the lemma. □

Theorem 3.6. *If θ is weakly inaccessible and $|\mathcal{P}_\kappa(\kappa^{+\alpha})| < \theta$ for all $\alpha < \theta$, then κ remains inaccessible in $V^{\mathbb{P}}$.*

Proof. It suffices to prove that κ remains regular. Let $p = (a, A) \in \mathbb{P}$, let $\delta < \kappa$, and suppose $\dot{f}$ is a $\mathbb{P}$-name forced by p to be a function from δ to κ. We will find $q \leq p$ forcing the range of $\dot{f}$ to be bounded below κ.

For all $\xi < \delta$, let D_ξ be the set of $(b, B) \in \mathbb{P}$ such that $(b, B) \Vdash$ "$\dot{f}(\xi) < \kappa_{b(\gamma^b)}$." Each D_ξ is a dense, open subset of $\mathbb{P}$. For $\xi < \delta$, let S_ξ be the set of stems b such that, for some B, $(b, B) \leq p$ and $(b, B) \in D_\xi$. For all $b \in S_\xi$, fix a B^ξ_b witnessing this. For each $\beta \in (\gamma^a, \theta)$, let $A^*(\beta)$ be the set of $y \in A(\beta)$ such that, for all $\xi < \delta$ and all $b \in S_\xi$ such that $b(\gamma^b) \prec y$ and $\gamma^b \in Z^\beta_y$, $y \in B^\xi_b(\beta)$. For $\beta \in \mathrm{dom}(A) \cap \gamma^a$, let $A^*(\beta) = A(\beta)$. Then $(a, A^*) \leq (a, A)$. Let R be the set of stems possible for (a, A^*). For $\gamma < \theta$, let $R_{<\gamma} = \{c \in R \mid \gamma^c < \gamma\}$. For all $c \in R$, let $p_c = c^\frown(a, A^*)$. For all $c \in R$ and $\xi < \delta$, apply Lemma 3.5 to p_c and D_ξ to obtain a sequence of trees $\langle T_{c,\xi,i} \mid i \leq |c|\rangle$. Let E be the set of $\gamma < \theta$ such that, for all $c \in T_{<\gamma}$ and all $\xi < \delta$, $\gamma_{T_{c,\xi,|c|}} < \gamma$. E is club in θ. Fix $\gamma \in E \setminus (\gamma^a + 1)$. By the discussion preceding Lemma 3.5, choose $z \in A^*(\gamma)$ such that, for all $c \in R_{<\gamma}$ such that $c(\gamma^c) \prec z$ and $\gamma^c \in Z^\gamma_z$ and for all $\xi < \delta$, $T_{c,\xi,|c|} \restriction (\gamma, z)$ is fat below (γ, z). Then $q = (a^\frown(\gamma, z), A^{**}) \leq (a, A^*)$, where $A^{**}(\alpha) = A^*(\alpha)$ for all $\alpha \in \mathrm{dom}(A^*) \cap \gamma^a$ and all $\alpha \in (\gamma, \theta)$, and $A^{**}(\alpha) = \{x \in A^*(\alpha) \mid x \prec z\}$ for all $\alpha \in (\gamma^a, \gamma)$.

We claim that q forces the range of $\dot{f}$ to be bounded below κ_z. Suppose for sake of contradiction that there is $\xi < \delta$ and $r \leq q$ such that $r \Vdash$ "$\dot{f}(\xi) \geq \kappa_z$." Let $r = (d, F)$, and let $c = \{(\alpha, x) \in d \mid \alpha < \gamma\}$. Then $c \in R_{<\gamma}$, $c(\gamma^c) \prec z$, and $\gamma^c \in Z^\gamma_z$, so $T_{c,\xi,|c|} \restriction (\gamma, z)$ is fat below (γ, z). Suppose that, for all $i \leq |c|$, n_i is the height of $T_{c,\xi,i}$. Then, for all $i \leq |c|$, we can find maximal elements $\langle(\beta^i_\ell, y^i_\ell) \mid \ell < n_i\rangle$ of $T_{c,\xi,i}$ such that, letting $c' = c \cup \{(\beta^i_\ell, y^i_\ell) \mid i \leq |c|, \ell < n_i\}$, $c' \cup d$ is possible for r. In particular, for all $(\alpha, x) \in c'$, $x \prec z$ and $\alpha \in Z^\gamma_z$, and so $\kappa_x < \kappa_z$. Also, there is B' such that $(c', B') \in D_\xi$ and, by our construction of A^*, we may assume that, for $\alpha \in \mathrm{dom}(B') \cap (\gamma^{c'}, \theta)$, $B'(\alpha) = A^*(\alpha)$. All of this together means that (c', B') and r are compatible. However, as $(c', B') \in D_\xi$, $(c', B') \Vdash$ "$\dot{f}(\xi) < \kappa_{c'(\gamma^{c'})} < \kappa_z$," contradicting the assumption that $r \Vdash$ "$\dot{f}(\xi) \geq \kappa_z$." □

The model for Theorem 1.1 will be the generic extension by $\mathbb{P}$ where θ is the least weakly inaccessible above κ. We note that by our preparation of the universe at the beginning of Section 2.1, we can assume that for every $\beta < \theta$ and every $y \in X_\beta$, 2^{κ_y} is at least the least inaccessible above κ_y. Now if G is $\mathbb{P}$-generic, it follows that for all $\nu \in C_G$, SCH fails at ν in $V[G]$.

4. Approachability

In this section we characterize precisely which successors ν^+ for $\nu \in C$ have reflection properties. For this section, assume that θ is the least weakly inaccessible cardinal above κ.

We make a definition which will be used frequently below.

Definition For $\beta < \theta$ and $y \in X_\beta$, we write $o(y)$ for $f^\beta_\beta(\kappa_y) = \mathrm{otp}(Z^\beta_y)$.

For $\nu < \kappa$, we let $\theta(\nu)$ be the least weakly inaccessible cardinal greater than ν. Using this notation we have $o(y) < \theta(\kappa_y)$.

Lemma 4.1. *$\mathbb{P}_{\vec{U}_y}$ as defined above Lemma 3.3 has the $\kappa_y^{+o(y)+1}$-Knaster property.*

Proof. It is not hard to see that there are just $\kappa_y^{+o(y)}$ many stems in this poset and that conditions with the same stem are compatible. □

If β is a limit ordinal and G_y is $\mathbb{P}_{\vec{U}_y}$-generic over V, then an easy genericity argument implies that $\bigcup C^{\mathrm{sc}}_{G_y} = \kappa_y^{+o(y)}$ and thus, in $V[G_y]$, $|\kappa_y^{+o(y)}| = \kappa_y$. Therefore, if G is $\mathbb{P}$-generic over V, then, for all $y \in C^{\mathrm{sc}}_G$ with $\kappa_y \in \lim(C_G)$, $(\kappa_y^+)^{V[G]} = (\kappa_y^{+o(y)+1})^V$. Moreover, if $\mathrm{cf}^V(o(y)) < \kappa_y$, then $\mathrm{cf}^{V[G]}(\kappa_y) = \mathrm{cf}^{V[G]}(o(y))$.

Lemma 4.2. *Suppose that $\beta < \theta$ is a limit ordinal, $y \in X_\beta$, $y' \in X_{\beta+1}$ and p is a condition such that $a^p(\beta) = y$ and $a^p(\beta + 1) = y'$. If μ is an ordinal of cofinality δ with $\kappa_y^{+o(y)+1} \leq \delta < \kappa_{y'}$ and $\dot{C}$ is a $\mathbb{P}$-name for a club subset of μ, then there are $p' \leq^* p$ and $D \subseteq \mu$ club such that p' forces $D \subseteq \dot{C}$.*

Proof. First we show that it is enough to consider $\delta = \mu$. Assume for the moment that $\delta < \mu$. Let $\pi : \delta \to \mu$ be an increasing, continuous and cofinal function. Now by passing to a name for a subset of $\dot{C}$ we can assume that it is forced that $\dot{C}$ is a subset of the range of π. Now a condition will force that there is a ground model club contained in $\dot{C}$ if and only if there is a ground model club contained in $\pi^{-1}(\dot{C})$.

So we may assume that $\delta = \mu$. By Lemma 3.3 and Lemma 3.4, there is a direct extension p' of p which forces $\dot{C}$ to be in the extension by $\mathbb{P}_{\vec{U}_y}$. Now by a standard argument using the $\kappa_y^{+o(y)+1}$-cc of $\mathbb{P}_{\vec{U}_y}$, there is a club $D \subseteq \mu$ in V such that p' forces $D \subseteq \dot{C}$. □

We use Lemma 4.2 to show that the approachability property fails at certain points along our Radin club.

Lemma 4.3. *Suppose that β is a limit ordinal with $\mathrm{cf}^V(\beta) < \kappa$ and p is a condition such that $a^p(\beta) = y$. Then p forces that $\kappa_y^{+o(y)+1} \notin I[\kappa_y^{+o(y)+1}]$.*

Proof. Assume for a contradiction that (some extension of) p forces $\kappa_y^{+o(y)+1} \in I[\kappa_y^{+o(y)+1}]$. Let $\langle \dot{z}_\gamma \mid \gamma < \kappa_y^{+o(y)+1} \rangle$ be a name for the approachability witness. We can assume that the order type of each $\dot{z}_\gamma$ is forced to be less than κ_y. By the Factorization Lemma, $\mathbb{P}_{\vec{U}}/p \cong \mathbb{P}_{\vec{U}_y}/p_0 \times \mathbb{P}_{\vec{U},\beta+1}/p_1$ for some p_0 and p_1. By the Prikry property, $\mathbb{P}_{\vec{U},\beta+1}/p_1$ does not add any new subsets to $\kappa_y^{+o(y)+1}$, so we may in fact assume that $\langle \dot{z}_\gamma \mid \gamma < \kappa_y^{+o(y)+1} \rangle$ is a $\mathbb{P}_{\vec{U}_y}$ name forced by p_0 to be a witness to approachability.

Fix $\alpha \in Z_y^\beta \setminus \max(\mathrm{dom}(a^p) \cap \beta)$. Let $j : V \to M$ witness that κ_y is $\kappa_y^{+o(y)+1}$-supercompact using an ultrapower by an ultrafilter which projects to $\bar{u}_\alpha^\beta(y)$. Let $\pi : y \cap \kappa^{+\alpha} \to \kappa_y^{+f_\alpha^\beta(\kappa_y)}$ be the unique order-preserving bijection. Let $\bar{A}^p(\alpha) = \{\pi``x \mid x \in A^p(\alpha)\}$. Note that $\bar{A}^p(\alpha) = a^{p_0}(f_\alpha^\beta(\kappa_y))$ and $\hat{y} := j``\kappa_y^{+f_\alpha^\beta(\kappa_y)} \in j(\bar{A}^p(\alpha))$.

Let $\hat{p} \leq j(p_0)$ be a condition in $j(\mathbb{P}_{\vec{U}_y})$ such that $a^{\hat{p}}(j(f_\alpha^\beta(\kappa_y))) = \hat{y}$ and $a^{\hat{p}}(j(f_\alpha^\beta(\kappa_y)) + 1) = \hat{y}'$ for some $\hat{y}'$. Set $\mu = \sup j``\kappa_y^{+o(y)+1}$ and $\delta = \mathrm{cf}(\mu)$. We have that $\hat{p}$ forces that μ is approachable with respect to $j(\langle \dot{z}_\gamma \mid \gamma < \kappa_y^{+o(y)+1} \rangle)$, so there is a name $\dot{C}$ for a club subset of μ such that, for all $\gamma < \mu$, $\hat{P}$ forces that $\dot{C} \cap \gamma$ is enumerated as $j(\dot{z})_{\gamma'}$ for some $\gamma' < \mu$.

By Lemma 4.2, there are a club $D \subseteq \mu$ in M and a direct extension $\hat{p}'$ of $\hat{p}$ such that $\hat{p}'$ forces $D \subseteq \dot{C}$. Let $E = \{\gamma < \kappa_y^{+o(y)+1} \mid j(\gamma) \in D\}$. It is straightforward to see that E is $< \kappa_y$-club in $\kappa_y^{+o(y)+1}$. Let γ^* be the $\kappa_y^{+o(y)}$-th element in an increasing enumeration of E. We can assume that there is an index $\hat{\gamma}$ such that $\hat{p}'$ forces that $\dot{C} \cap j(\gamma^*)$ is enumerated before stage $j(\hat{\gamma})$.

Note that $\mathrm{cf}(o(y)) < \kappa_y$ since $\mathrm{cf}(\beta) < \kappa_y$. Now if $x \subseteq E \cap \gamma^*$ has order type at most $\mathrm{cf}(o(y))$, then there is a condition $p_x \leq p_0$ which forces $x \subseteq \dot{z}_\gamma$ for some $\gamma < \hat{\gamma}$. To see this notice that j of this statement is witnessed by $\hat{p}'$.

By the chain condition of $\mathbb{P}_{\vec{U}_y}$, we can find a condition which forces that for $\kappa_y^{+o(y)+1}$ many x, p_x is in the generic filter. This is impossible, since each $\dot{z}_\gamma$ is forced to have order type less than κ_y and hence, in the extension, $|\bigcup_{\gamma<\hat{\gamma}} \mathcal{P}(\dot{z}_\gamma)| \leq \kappa_y$. □

Next, we show that weak square holds at points taken from X_β where $\mathrm{cf}(\beta) \geq \kappa$. To do so, we need a lemma about the cofinalities of points in the extension and a few definitions.

Let G be $\mathbb{P}$-generic over V. Suppose that $\nu = \kappa \cap x \in C$, with $x \in C_G^{\mathrm{sc}}$. Let $p = (a, A) \in G$ be such that $x = a(\beta)$ for some $\beta \in \mathrm{dom}(a)$. Recall that, by the definition of the sets X_β, $\beta < \theta$ and $\mathbb{P}$, we have:

- β is a limit ordinal if and only of $o(x)$ is;
- if β is limit, then $(\nu^+)^{V[G]} = (\nu^{+o(x)+1})^V$;
- $\mathrm{cf}(\beta) \geq \kappa$ if and only if $\mathrm{cf}(o(x)) \geq \nu$.

Lemma 4.4. *If β is limit and $\operatorname{cf}(\beta) \geq \kappa$, then ν and $\nu^{+o(x)}$ change their cofinality to ω in $V[G]$.*

Proof. Work in V. Using the Factorization Lemma, $\mathbb{P}_{\vec{U}}/p \cong \mathbb{P}_{\vec{U}_x}/p_0 \times \mathbb{P}_{\vec{U},\beta+1}/p_1$ for some p_0 and p_1. As $\mathbb{P}_{\vec{U},\beta+1}/p_1$ does not add new bounded subsets to $\theta(\nu)$, it is sufficient to focus on the forcing $\mathbb{P}_{\vec{U}_x}$ which adds a Radin club to ν. For notational simplicity, let $\delta = f^{\beta}_{\beta}(x) = o(x)$, and let $\vec{U}_x = \langle V_\xi \mid \xi < \delta\rangle$. We have $\rho := \operatorname{cf}(\delta) \geq \nu$ and $\delta < \theta(\nu)$. We will show that ν and $\nu^{+\delta}$ change their cofinalities to ω after forcing with $\mathbb{P}_{\vec{U}_x}$.

Choose an increasing continuous sequence $\vec{\delta} = \langle \delta_\alpha \mid \alpha < \rho\rangle$ cofinal in δ. Let G_x be $\mathbb{P}_{\vec{U}_x}$-generic over V. For every $y \in C^{\mathrm{sc}}_{G_x}$, let $\alpha(y) < \rho$ be the minimal $\alpha < \rho$ so that $y = b(\beta')$ for some $\beta' \leq \beta_\alpha$ and some $q = (b, B) \in G_x$.

Since $\delta < \theta(\nu)$, $\nu^{+\delta} > \delta \geq \rho$. Let $\alpha_0 < \rho$ be the least ordinal so that $\rho < \nu^{+\delta_{\alpha_0}}$. By reindexing, we can assume that $\alpha_0 = 0$. Note that, for every δ' with $\delta_0 \leq \delta' < \delta$, $Y_{\delta'} = \{y \in X_{\beta'} \mid \alpha(y) \in y\}$ belongs to $V_{\delta'} \subset \mathcal{P}(\mathcal{P}_\nu(\nu^{+\delta'}))$.

It follows that, in $V[G_x]$, there is some $\nu_0 \in C_{G_x}$ such that, for every $y \in C^{\mathrm{sc}}_{G_x}$, if $y = a(\delta')$ for some $\delta' \geq \delta_0$ and $\nu_y := y \cap \nu > \nu_0$, then $y \in Y_{\delta'}$. Let y_0 be the minimal $y \in C^{\mathrm{sc}}_{G_x}$ satisfying the above. Starting from y_0, we define a sequence $\vec{y} = \langle y_n \mid n < \omega\rangle \subset C^{\mathrm{sc}}_{G_x}$. For each $n < \omega$, let y_{n+1} be the minimal y above y_n in $C^{\mathrm{sc}}_{G_x}$ so that $\alpha(y) > \sup(y_n \cap \rho) < \rho$. Let $\nu_\omega = \bigcup_{n<\omega} \nu_{y_n}$. We claim that $\nu_\omega = \nu$. Suppose otherwise. Then $\nu_\omega = \nu_y$ for some $y \in C^{\mathrm{sc}}_{G_x}$.

Choose $q = (b, B) \in G_x$ and $\delta' \geq \delta_0$ such that $y = b(\delta')$. Let $\alpha = \alpha(y) < \rho$. Then $\alpha \in y < \sup(y \cap \rho)$. Since $\langle \nu_{y_n} \mid n < \omega\rangle$ is cofinal in κ_ω, we have $y \cap \rho = \bigcup_{n<\omega}(y_n \cap \rho)$. There is thus some $m < \omega$ such that $\sup(y_m \cap \rho) > \alpha$. But $\alpha \geq \alpha(y_{m+1})$, which means that $\alpha(y_{m+1}) < \sup(y_m \cap \rho)$, contradicting the definition of the sequence $\vec{y}$.

It follows that $\nu = \nu_\omega$, so ν changes its cofinality to ω. The set $C^{\mathrm{sc}}_{G_x} \subset \mathcal{P}_\nu(\nu^{+\delta})$ is $\subseteq$-cofinal in $\mathcal{P}_\nu(\nu^{+\delta})$. Since $\langle \nu_{y_n} \mid n < \omega\rangle$ is cofinal in ν, $\langle y_n \mid n < \omega\rangle$ is $\subset$-cofinal in $C^{\mathrm{sc}}_{G_x}$. Thus, $\nu^{+\delta} = \bigcup_{n<\omega} y_n$. It follows that $\operatorname{cf}(\nu^{\delta}) = \omega$, as each y_n is bounded in $\nu^{+\delta}$. □

It follows easily that regular cardinals between ν and $\nu^{+o(\nu)}$ also change their cofinality to ω. To show that weak square holds, we need the definition of a partial square sequence.

Definition Let $\lambda < \delta$ be regular cardinals, and let $S \subseteq \delta \cap \operatorname{cof}(\lambda)$. We say that S *carries a partial square sequence* if there is a sequence $\langle C_\gamma \mid \gamma \in S\rangle$ such that:

1. for all $\gamma \in S$, C_γ is club in γ and $\operatorname{otp}(C_\gamma) = \lambda$;
2. for all $\gamma < \gamma^*$ from S, if β is a limit point of C_γ and C_{γ^*}, then $C_\gamma \cap \beta = C_{\gamma^*} \cap \beta$.

Next, we need a theorem of Dzamonja and Shelah [5].

Theorem 4.5. *Let λ be a regular cardinal and $\mu > \lambda$ be singular. If $\operatorname{cf}([\mu]^{\leq\lambda}, \subseteq) = \mu$, then $\mu^+ \cap \operatorname{cof}(\leq \lambda)$ is the union of μ many sets, each of which carries a partial square sequence.*

Lemma 4.6. *Suppose that $\beta \in \theta \cap \operatorname{cof}(\geq \kappa)$ and $p \in G$ is a condition such that $a^p(\beta) = y$ for some y. Then, in $V[G]$, $\square^*_{\kappa_y}$ holds.*

Proof. Work in V. For each regular $\lambda < \kappa_y$, we have that $(\kappa_y^{+o(y)})^{\leq\lambda} = \kappa_y^{+o(y)}$ using the supercompactness of κ_y, and hence $\operatorname{cf}([\kappa_y^{+o(y)}]^{\leq\lambda}, \subseteq) = \kappa_y^{+o(y)}$. Therefore, by

Theorem 4.5, we can write $\kappa_y^{+o(y)+1} \cap \mathrm{cof}(\leq \lambda)$ as the union of $\kappa_y^{+o(y)}$ sets which have partial squares. We call these sequences $\vec{C}^{\lambda,i}$ for $i < \kappa_y^{+o(y)}$.

By Lemma 4.4, in $V[G]$, we have that each cardinal in the interval $[\kappa_y, \kappa_y^{+o(y)}]$ changes its cofinality to ω. So, in $V[G]$, we can write $\kappa_y^{+o(y)+1}$ as the disjoint union of $(\kappa_y^{+o(y)+1} \cap \mathrm{cof}(< \kappa_y))^V$, which we call T_0, and a set T_1 of ordinals of countable cofinality.

We define a weak square sequence as follows. For $\gamma \in T_0$, we let $\mathcal{C}_\gamma = \{C^{\lambda,i}_{\gamma'} \cap \gamma \mid \lambda < \kappa_y,\ i < \kappa_y^{+o(y)}$ and γ is a limit point of $C^{\lambda,i}_{\gamma'}\}$. For $\gamma \in T_1$, we let $\mathcal{C}_\gamma = \{C\}$ where C is some cofinal ω-sequence in γ.

The coherence is obvious, so we just have to check that each $\mathcal{C}_\gamma$ is not too large. Suppose that there is γ such that $|\mathcal{C}_\gamma| \geq \kappa_y^{+o(y)+1}$. Then, by the pigeonhole principle, we can find two elements C and C' on which the indices λ and i are the same. But then we have that $C = C'$ by the coherence of the partial square sequence with indices λ and i, which is a contradiction. $\square$

We are now ready to complete the proof of Theorem 1.1. In $V[G]$, let S be the set of singular cardinals $\nu \in C_G$ such that $\square^*_\nu$ holds. By Lemmas 4.3 and 4.4, $S \subseteq \kappa \cap \mathrm{cof}(\omega)$. By Lemma 4.6 and genericity, S is stationary in κ. However, we claim that S can be made non-stationary in a cofinality-preserving forcing extension of $V[G]$.

Lemma 4.7. *Let $\delta \in \lim(C_G) \cap \mathrm{cof}(> \omega)$. In $V[G]$, $S \cap \delta$ is non-stationary in δ.*

Proof. Fix $p = (a, A) \in G$ such that, for some $\beta \in (\theta \cap \mathrm{cof}(< \kappa))^V$, $a(\beta) = y$, where $\kappa_y = \delta$. Work in V, letting $\dot{S}$ be a canonical $\mathbb{P}_{\vec{U}}$-name for S. We will find $q \leq p$ such that $q \Vdash$ "$\dot{S} \cap \delta$ is non-stationary."

Let $\mu = \mathrm{cf}(\beta)$. Since $\mu < \kappa$ and $y \in X_\beta$, we have that $\mu < \kappa_y$ and Z^β_y is $< \kappa_y$-closed and unbounded in β. Find $D \subseteq Z^\beta_y$ such that:

- D is club in β;
- $\mathrm{otp}(D) = \mu$;
- $\min(D) > \max(\mathrm{dom}(a) \cap \beta)$.

For each $\alpha \in Z^\beta_y \setminus \min(D)$, let $Y_\alpha = \{x \in \mathcal{P}_\delta(y \cap \kappa^{+\alpha}) \mid D \cap \alpha \subseteq Z^\alpha_x\}$, and note that $Y_\alpha \in u^\beta_\alpha(y)$. Define $q = (a, B) \leq p$ by letting $B(\alpha) = A(\alpha) \cap Y_\alpha$ for $\alpha \in Z^\beta_y \setminus \min(D)$ and $B(\alpha) = A(\alpha)$ for all other values of α. Now $q \Vdash$ "$D \subseteq \dot{C}_G$." Let $\dot{E}$ be a $\mathbb{P}_{\vec{U}}/q$-name for $\{\kappa_x \mid$ for some $r \in G$ and $\alpha \in D$, $r(\alpha) = x\}$. $q \Vdash$ "$\dot{E}$ is club in δ" and, since $\lim(D) \subseteq \mathrm{cof}(< \kappa)$, Lemma 4.3 implies that $q \Vdash$ "$\lim(\dot{E}) \cap \dot{S} = \emptyset$.". $\square$

In particular, in $V[G]$, $\kappa \setminus S$ is a fat stationary set. Let $\mathbb{Q}$ be the forcing notion whose conditions are closed, bounded subsets of κ disjoint from S, ordered by end-extension. $\mathbb{Q}$ adds a club in κ disjoint from S and, by a result of Abraham and Shelah from [2], $\mathbb{Q}$ is κ-distributive. Thus, if H is $\mathbb{Q}$-generic over $V[G]$, D is the generic club added by Q, and $E = D \cap C_G$, then E witnesses that $V[G * H]$ satisfies the conclusion of Theorem 1.1. Moreover, if we let $N = (V[G * H])_\kappa = V_\kappa^{V[G*H]}$, then N is a model of GB (Gödel-Bernays) with a class club E through its cardinals such that, for every $\nu \in E$, ν is a singular cardinal, SCH fails at ν, and $\square^*_\nu$ fails.

5. Conclusion

In a forthcoming paper of the third author [23], a model is constructed in which $\aleph_{\omega^2}$ is strong limit and weak square fails for all cardinals in the interval $[\aleph_1, \aleph_{\omega^2+2}]$. In particular, it is shown that one can put collapses between the Prikry points of the Gitik-Sharon [10] construction which will make κ into $\aleph_{\omega^2}$ and enforce the failure of weak square below $\aleph_{\omega^2}$.

It is reasonable to believe that this construction could be combined with the forcing from Theorem 1.1, but we are left with the unsatisfactory result that weak square will hold at some successors of singulars in the extension. To make this precise, we formulate a question which seems to capture the limit of a naive combination of the two techniques.

Question 5.1. *Suppose that κ is a singular cardinal of cofinality ω such that $\square^*_\lambda$ fails for all $\lambda \in [\aleph_1, \kappa)$ and $|\{\lambda < \kappa \mid \lambda \text{ is singular strong limit }\}| = \kappa$. Is there a $\square^*_\kappa$-sequence?*

We also ask two other natural questions.

Question 5.2. *Is there a version of Theorem 1.1 in which the failure of $\square^*_\nu$ is replaced with the tree property at ν^+?*

Question 5.3. *Let $C_G \subset \kappa$ be a generic Radin club added by the poset $\mathbb{P}$ defined in Section 3. Does $\square_{\nu,\omega}$ fail at every ordinal $\nu \in C_G$?*

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