COMPACTNESS FOR THE CHROMATIC NUMBER AT $\aleph_{\omega_1+1}$

SPENCER UNGER

In this note we prove the following theorem.

Theorem 1. *If κ is a supercompact cardinal, then there is a forcing extension in which $\kappa = \aleph_{\omega_1}$ and if $0 < \alpha < \omega_1$ and G is a graph of size $\aleph_{\omega_1+1}$ each of whose subgraphs of size less than $\aleph_{\omega_1}$ has chromatic number at most $\aleph_{\alpha+1}$, G has chromatic number atmost $\aleph_{\alpha+1}$.*

Ben-David and Magidor showed that in a model of Magidor [3] there is a highly indecomposable ultrafilter on $\aleph_{\omega+1}$. Using this model, Shelah [4] showed the following:

Theorem 2. *In the model of Magidor, if $0 < n < \omega$ and G is a graph of size $\aleph_{\omega+1}$ all of whose subgraphs of size less than $\aleph_\omega$ have chromatic number at most $\aleph_n$, then G has chromatic number at most $\aleph_n$.*

In this paper we produce a higher version of Magidor's model [3] where our large cardinal κ becomes $\aleph_{\omega_1}$ and we have the conclusion of Theorem 1. The techniques here are standard and so we will not give any proofs of the basic properties of our forcing.

We note that there is a minor omission in the proof of the main theorem of Ben-David and Magidor relating to a claim about the homogeneity of the forcing in [3]. The argument there requires some further diagonalization. We change our forcing slightly to obtain a more full homogeneity property, which avoids this extra diagonalization.

1. The forcing

We work in a model V of GCH with a supercompact cardinal κ. Let $\langle U_{\zeta,\alpha} \mid \zeta \leq \kappa, \alpha < o(\zeta)\rangle$ be a coherent sequence of supercompactness measures in the sense of Krueger's [2] where each $U_{\zeta,\alpha}$ is a measure on $\mathcal{P}_\zeta(\zeta^+)$ and $o(\kappa) = \omega_1$. The precise definition is not important for us, but we will need a few of its consequences. For the measures $U_{\kappa,\alpha}$ we will just drop the κ and write U_α.

For each $\alpha < \beta < \omega_1$ let $x \mapsto U^\beta_\alpha(x)$ be the map whose equivalence class modulo U_β is U_α. By the coherence of our original sequence we can just take $U^\beta_\alpha(x)$ to be $U_{\kappa_x,\alpha}$. There is a natural "uncollapse" $\hat{U}^\beta_\alpha(x)$ of $U^\beta_\alpha(x)$ along the inverse of the transitive collapse of x. We also "lift" this notation to the ultrapower and write $\hat{U}^\beta_\alpha$ for the uncollapse of U_α along the inverse of the transitive collapse of j_{U_β}"κ^+. For the definition of the forcing it will be convenient to have a names for all of

Date: December 1, 2015.

The motivation for Theorem 1 came from a series of tutorials given by Peter Komjath at the 2015 Young Set Theory meeting at the Hebrew University. We would like to thank Peter for asking the question and for his excellent series of tutorials.

these measures despite the fact that they are all isomorphic to measures from our original coherent sequence.

We need a sequence of claims to show that these functions have some coherence properties.

Claim 3. *Let $\alpha < \beta < \omega_1$. For any $A \in U_\alpha$ there is a U_β-measure one set of y such that $\{x \in A \mid x \prec y\} \in \hat{U}^\beta_\alpha(y)$.*

For convenience we will write $A \upharpoonright y$ for the set $\{x \in A \mid x \prec y\}$.

Claim 4. *Let $\alpha < \beta < \gamma < \omega_1$. There is a U_γ measure one set of z such that for all $A \in \hat{U}^\gamma_\alpha(z)$, there is a $\hat{U}^\gamma_\beta(z)$-measure one set of y such that $A \upharpoonright y \in \hat{U}^\beta_\alpha(y)$.*

This is just a relative of the previous claim but in the ultrapower by U_γ. From the previous two claims we could define a forcing which collapses κ to be $\aleph_{\omega_1}$, but we make some further definitions to achieve a stronger homogeneity property.

Claim 5. *For $\zeta \leq \kappa$ and $\alpha < o(\zeta)$, there is generic filter $G_{\zeta,\alpha}$ for $\mathrm{Coll}(\kappa^{++}, < j_{U_{\zeta,\alpha}}(\kappa))$ as computed in $\mathrm{Ult}(V, U_{\zeta,\alpha})$ such that the $G_{\zeta,\alpha}$ are coherent in the sense that for all ζ and $\alpha < o(\zeta)$, $[x \mapsto \langle G_{\kappa_x,\beta} \mid \beta < \alpha\rangle]_{U_{\zeta,\alpha}} = \langle G_{\zeta,\beta} \mid \beta < \alpha\rangle$.*

As for the ultrafilters, we write G_α for $G_{\kappa,\alpha}$. Let $x \mapsto G^\beta_\alpha(x)$ be the function that represents G_α in the ultrapower by U_β. As with the ultrafilters we can take each $G^\beta_\alpha(x)$ to be equal to $G_{\kappa_x,\alpha}$. There is a natural uncollapsed version of $G^\beta_\alpha(x)$ which we call $\hat{G}^\beta_\alpha(x)$ and we extend this notation to the ultrapower by U_β by writing $\hat{G}^\beta_\alpha$ for the uncollapse of G_α along j_{U_β}"κ^+. We now need the analogs of Claims 3 and 4, but for functions which represent elements of G_α and $G^\beta_\alpha(x)$.

Claim 6. *Let $\alpha < \beta < \omega_1$. For any F with $[F] \in G_\alpha$, there is a U_β-measure one set of y such that the equivalence class of $F \upharpoonright \{x \mid x \prec y\}$ is in $\hat{G}^\beta_\alpha(y)$.*

For convenience we write $F \upharpoonright y$ for $F \upharpoonright \{x \mid x \prec y\}$.

Claim 7. *Let $\alpha < \beta < \gamma < \omega_1$. There is a U_γ-measure one set of z such that for all F with $[F] \in \hat{G}^\gamma_\alpha$, there is a $\hat{U}^\gamma_\beta(z)$ measure one set of y such that $[F \upharpoonright y] \in \hat{G}^\beta_\alpha$.*

Again this claim is a relative of the previous one, but in the ultrapower by U_γ. For each $\gamma < \omega_1$, we let $Z^\gamma \in U_\gamma$ be a set on which the Claims 4 and 7 hold for all choices of α and β. We can assume that for every $z \in Z^\gamma$, $z \cap \kappa \in \kappa$ is inaccessible and $\mathrm{ot}(z) = (z \cap \kappa)^+$. We will write κ_z for $z \cap \kappa$.

We are now ready to define the forcing $\mathbb{P}$. Conditions are of the form (f, H) where

1. f is a function and $\mathrm{dom}(f)$ is a finite subset of ω_1 containing 0.
2. For all $\alpha \in \mathrm{dom}(f)$, $f(\alpha)$ is a pair (x_α, c_α) such that $x_\alpha \in Z^\alpha$ and $c_\alpha \in \mathrm{Coll}(\kappa^{++}_{x_\alpha}, < \gamma)$ where $\gamma = \kappa_{x_\beta}$ if $\beta = \min(\mathrm{dom}(f) \setminus \alpha + 1)$ if it exists and $\gamma = \kappa$ otherwise. For $\alpha = 0$ we assume that $x_0 = \omega$.
3. For all $\alpha < \beta$ from $\mathrm{dom}(f)$, $x_\alpha \prec x_\beta$ and $\kappa_{x_\beta} > \max(\mathrm{ran}(c_\alpha))$.
4. H is a function and $\mathrm{dom}(H) = \omega_1 \setminus \mathrm{dom}(f)$.
5. For all $\alpha \in \mathrm{dom}(H)$, if $\alpha < \max(\mathrm{dom}(f))$, then $[H(\alpha)] \in \hat{G}^\beta_\alpha(x_\beta)$ where $\beta = \min(\mathrm{dom}(f) \setminus \alpha + 1)$ and if $\alpha > \max(\mathrm{dom}(f))$, then $[H(\alpha)] \in G_\alpha$.

For a condition $p \in \mathbb{P}$ we write $p = (f^p, H^p)$ and further x^p_α, c^p_α, etc. For the ordering we set $p \leq q$ if:

(1) $\mathrm{dom}(f^p) \supseteq \mathrm{dom}(f^q)$.
(2) For all $\alpha \in \mathrm{dom}(f^q)$, $x^p_\alpha = x^q_\alpha$ and $c^p_\alpha \leq c^q_\alpha$
(3) For $\alpha \in \mathrm{dom}(f^p) \setminus \mathrm{dom}(f^q)$, $x^p_\alpha \in \mathrm{dom}(H^q(\alpha))$ and $c^p_\alpha \leq H^q(\alpha)(x^p_\alpha)$.
(4) For all $\alpha \in \mathrm{dom}(H^p)$, $\mathrm{dom}(H^p(\alpha)) \subseteq \mathrm{dom}(H^q(\alpha))$ and for all $x \in \mathrm{dom}(H^p(\alpha))$, $H^p(\alpha)(x) \leq H^q(\alpha)(x)$.

It is straightforward to see that that the ordering is transitive. We define $p \leq^* q$ if $p \leq q$ and $\mathrm{dom}(f^p) = \mathrm{dom}(f^q)$. The following lemma is standard in the theory of Prikry type forcings with interleaved collapses.

Lemma 8. *For every formula φ in the forcing language and condition $p \in \mathbb{P}$, there is $q \leq^* p$ such that q decides φ.*

We can define some natural restrictions of the forcing where we restrict each function to have domain α for some $\alpha < \omega_1$. We call this poset $\mathbb{P}_\alpha$ and note that there is a projection from $\mathbb{P}$ to $\mathbb{P}_\alpha$. As a collorary of the previous lemma we have the following.

Corollary 9. *If $\dot{X}$ is a $\mathbb{P}$-name for a bounded subset of κ, then it is forced that the interpretation of $\dot{X}$ is in the extension by $\mathbb{P}_\alpha$ for some α.*

It follows that in the generic extension $\kappa = \aleph_{\omega_1}$ and GCH holds below $\aleph_{\omega_1}$.

Remark 10. *We note that for two conditions p and q if $f^p = f^q$, then p and q are compatible. This is not true of Magidor's poset from* [3].

Note that by an easy density argument if $\langle x_\alpha \mid \alpha < \omega_1 \rangle$ is the generic sequence of elements of $\mathcal{P}_\kappa(\kappa^+)$ added by $\mathbb{P}$, then $\bigcup_{\alpha<\omega_1} x_\alpha = \kappa^+$. It follows that κ^+ is collapsed in the extension by $\mathbb{P}$.

For $\alpha < \omega_1$, we let $\kappa_\alpha = \kappa_{x_\alpha}$. We define an inner model of the extension by $\mathbb{P}$ which will be a model for Theorem 1. Let $W = V[\langle \kappa_\alpha \mid \alpha < \omega_1 \rangle, \langle C_\alpha \mid \alpha < \omega_1 \rangle]$ where C_α is the $\mathrm{Coll}(\kappa^{++}_{x_\alpha}, < \kappa_{x_{\alpha+1}})$ generic added by $\mathbb{P}$. Clearly $V \subseteq W \subseteq V[\mathbb{P}]$.

Before we give the argument for the chromatic number, we need a claim about automorphisms of $\mathbb{P}$. Suppose that we have two conditions $p, q \in \mathbb{P}$ such that $\mathrm{dom}(f^p) = \mathrm{dom}(f^q)$ and for all $\alpha \in \mathrm{dom}(f^p)$, $\kappa_{x^p_\alpha} = \kappa_{x^q_\alpha}$ and $c^p_\alpha = c^q_\alpha$.

Note that for $\alpha \in \mathrm{dom}(H^p)$ with $\alpha < \max(\mathrm{dom}(f^p))$, we have $\hat{U}^\beta_\alpha(x^p_\beta)$ is isomorphic to $\hat{U}^\beta_\alpha(x^q_\beta)$ for $\beta = \min(\mathrm{dom}(f^p)) \setminus \alpha$, since they both collapse to be $U_{\kappa_x,\alpha}$. ($U_{\kappa_x,\alpha}$ was a measure on the original coherent sequence.) Further there is a bijection Γ_α from x^p_α to x^q_α which fixes $\kappa_{x^p_\alpha}$ witnessing the isomorphism in the following sense. For $A \subseteq \mathcal{P}_{\kappa_{x^p_\alpha}}(x^p_\alpha)$, $A \in U^\beta_\alpha(x^p_\alpha)$ if and only if $\{\Gamma_\alpha “x \mid x \in A\} \in U^\beta_\alpha(x^q_\alpha)$. Further, we have for all functions F, $[F] \in G^\beta_\alpha(x^p_\alpha)$ if and only if $[F \circ \Gamma^{-1}_\alpha] \in G^\beta_\alpha(x^q_\alpha)$.

Now we can take Γ to be an automorphism of κ^+ which fixes κ such that $\bigcup_{\alpha \in \mathrm{dom}(f^p)} \Gamma_\alpha \subseteq \Gamma$. In fact we can take Γ to be the identity above some $\rho < \kappa^+$. It is now straightforward to see that this Γ induces an automorphism Γ^* of $\mathbb{P}$ such that $\Gamma^*(p)$ is compatible with q.

We are now ready to argue for the compactness of the chromatic number. The arguments given below were inspired by a talk given by Menachem Magidor on compactness for chromatic numbers. In the extension by $\mathbb{P}$, we define a filter $\mathcal{F}$ on $\mathcal{P}_{\aleph_{\omega_1}}(\aleph_{\omega_1+1})^W$ by setting $Y \in \mathcal{F}$ if and only if there is an α such that for all $\beta > \alpha$, $x_\beta \in Y$. Clearly $\mathcal{F}$ is a filter and for all $\gamma < \aleph_{\omega_1+1}$, $\{x \in W \mid \gamma \in x\} \in \mathcal{F}$.

Claim 11. *$\mathcal{F}$ is definable in W.*

Proof. Let $\dot{Y}$ be a $\mathbb{P}$-name for a subset of $\mathcal{P}_{\aleph_{\omega_1}}(\aleph_{\omega_1+1})^W$ whose interpretation is in W. We can assume that $\dot{Y}$ is fixed by any automorphism which fixes the objects generating W.

Suppose that $p, q \in \mathbb{P}$ such that $\mathrm{dom}(f^p) = \mathrm{dom}(f^q)$ and for all $\alpha \in \mathrm{dom}(f^p)$, $\kappa_{x^p_\alpha} = \kappa_{x^q_\alpha}$ and $c^p_\alpha = c^q_\alpha$. By our work with automorphisms above there is a Γ^* such that $\Gamma^*(p)$ is compatible with q. Moreover, Γ^* fixes any name $\dot{\mathcal{F}}$ for $\mathcal{F}$, since it cannot change names for a tail end of the x_α. It follows that p forces "$\dot{Y} \in \dot{\mathcal{F}}$" if and only q does and similarly for "$\dot{Y} \notin \dot{\mathcal{F}}$".

So we can define $\mathcal{F}$ in W as the set of Y such that for any invariant name $\dot{Y}$ for Y there is a condition $p \in \mathcal{P}$ such that for all $\alpha \in \mathrm{dom}(f^p)$, $\kappa_{x^p_\alpha} = \kappa_\alpha$ and $c^p_\alpha \in C_\alpha$ and $p \Vdash \dot{Y} \in \dot{\mathcal{F}}$. □

Let U in W be an ultrafilter extending $\mathcal{F} \cap W$. We have the following claim which essentially Lemma 2 from [1].

Claim 12. *If η is a regular cardinal with $\aleph_1 < \eta < \aleph_{\omega_1}$, then $|\mathrm{Ult}(\eta, U)| = \eta$.*

Proof. For each $f : \mathcal{P}_{\aleph_{\omega_1}}(\aleph_{\omega_1+1})^W \to \eta$ from W we can associate a function in $V[\mathbb{P}]$ g_f with domain ω_1 such that $g_f(\alpha) = f(x_\alpha)$. Now $[f]_U = [f']_U$ if and only if g_f and $g_{f'}$ are equal on a tail end. So since $\eta^{\omega_1} = \eta$ in $V[\mathbb{P}]$, we have that $\mathrm{Ult}(\eta, U)$ has size η. □

We can now argue for the compactness of the chromatic number in W.

Claim 13. *If η is a regular cardinal with $\aleph_1 < \eta < \aleph_{\omega_1}$ and G is a graph of size $\aleph_{\omega_1+1}$ all of whose subgraphs of size less than $\aleph_{\omega_1}$ have chromatic number at most η, then G has chromatic number at most η.*

Proof. We can assume that the set of vertices of the graph is $\aleph_{\omega_1+1}$. For each $x \in \mathcal{P}_{\aleph_{\omega_1}}(\aleph_{\omega_1+1})^W$ we have a function $\chi_x : x \to \eta$ which is a coloring of the graph restricted to x. Now for each $\beta < \aleph_{\omega_1+1}$, we define a function $f_\beta(x) = \chi_x(\beta)$. Each f_β is defined on a U-measure one set. We claim that the map $\beta \mapsto [f_\beta]_U$ is a coloring of G. There are at most η colors by the previous claim. Now if $\beta_0 < \beta_1$ have an edge in G, then for all x such that $\beta_0, \beta_1 \in x$, $\chi_x(\beta_0) \neq \chi_x(\beta_1)$. It follows that $[f_{\beta_0}]_U \neq [f_{\beta_1}]_U$ and this finishes the proof. □

References

1. Shai Ben-David and Menachem Magidor, *The weak $\square^*$ is really weaker than the full $\square$*, J. Symbolic Logic **51** (1986), no. 4, 1029–1033. MR 865928 (88a:03117)
2. John Krueger, *Radin forcing and its iterations*, Arch. Math. Logic **46** (2007), no. 3-4, 223–252. MR 2306177 (2009b:03138)
3. Menachem Magidor, *On the singular cardinals problem. I*, Israel J. Math. **28** (1977), no. 1-2, 1–31. MR 0491183 (58 #10449a)
4. Saharon Shelah, *Incompactness for chromatic numbers of graphs*, A tribute to Paul Erdős, Cambridge Univ. Press, Cambridge, 1990, pp. 361–371. MR 1117029 (92j:03045)