

SCALES AT \aleph_ω

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In this paper we analyze the PCF structure of a generic extension by the main forcing from our previous paper [19]. Our analysis relies on a different preparation than was used in the first paper. As a corollary of our analysis we have a partial answer the following question of Woodin: “Is it consistent that both the singular cardinals hypothesis and weak square fail at \aleph_ω ?” In particular we prove the following theorem:

Theorem 0.1. *It is consistent relative to a supercompact cardinal that \aleph_ω is a strong limit, $2^{\aleph_\omega} = \aleph_{\omega+2}$ and $\square_{\aleph_\omega, \aleph_n}$ fails for all $n < \omega$.*

The motivation for Woodin’s question comes from a tension between the failure of weak square and the failure of the singular cardinals hypothesis.

Definition 0.2. *Let ν be a singular cardinal. The singular cardinal hypothesis (SCH) at ν is the assertion “If ν is strong limit, then $2^\nu = \nu^+$.”*

We note that the singular cardinals hypothesis holds at every singular cardinal in Easton’s [5] models for possible behavior of the continuum function on regular cardinals. In particular the behavior of the continuum function on singular cardinals is determined by its restriction to regular cardinals. So models for the failure of SCH give non-trivial examples of possible behavior for the continuum function on singular cardinals.

The principle weak square was introduced by Jensen [9] in his study of the fine structure of L . Intermediate principles were introduced by Schimmerling [15] and it is these intermediate principles that we define here:

Definition 0.3. *Let ν and λ be cardinals. A $\square_{\nu, \lambda}$ -sequence is a sequence $\langle \mathcal{C}_\alpha \mid \alpha < \nu^+ \rangle$ such that*

- (1) *for all $\alpha < \nu^+$, $1 \leq |\mathcal{C}_\alpha| \leq \lambda$,*
- (2) *for all $\alpha < \nu^+$ and for all $C \in \mathcal{C}_\alpha$, C is club in α and $\text{o.t.}(C) \leq \nu$ and*
- (3) *for all $\alpha < \nu^+$, all $C \in \mathcal{C}_\alpha$ and all $\beta \in \lim(C)$, $C \cap \beta \in \mathcal{C}_\beta$.*

Jensen’s \square_ν^* is $\square_{\nu, \nu}$ under the above definition. The theorem which connects the combinatorial principle weak square to classical set theoretic objects is due to Jensen:

Theorem 0.4. *\square_ν^* holds if and only if there is a special ν^+ -Aronszajn tree.*

When ν is regular, an easy argument shows that \square_ν^* follows from the cardinal arithmetic $\nu^{<\nu} = \nu$. So in particular the successor of every inaccessible cardinal carries a special Aronszajn tree.

This brings us to the motivation for Woodin's question. The original model for the failure of the singular cardinals hypothesis is obtained as follows:

- (1) Start with a large cardinal κ and force $2^\kappa > \kappa^+$ while preserving the measurability of κ .
- (2) Force with Prikry forcing [14] to make κ singular strong limit of cofinality ω while preserving cardinals above κ .

By our remark above that the successor of an inaccessible cardinal has a special Aronszajn tree, we have that there is a special κ^+ -Aronszajn tree in the ground model for such a construction. Since special Aronszajn trees are preserved in cardinal preserving extensions, the tree from the ground model remains special in the extension and hence \square_κ^* holds. So initially it looked like forcing the failure of SCH would always give a weak square even for a typical large singular cardinal. It is worth noting that Cummings and Schimmerling [4] later showed that Prikry forcing at κ adds a $\square_{\kappa,\omega}$ -sequence.

Part of the motivation for Woodin's question (when formulated for an arbitrary singular cardinal) was to test whether the above method for obtaining the failure of the singular cardinals hypothesis was in fact the only such method. It turned out that there were other methods for obtaining the failure of SCH, but they too gave weak square. One notable instance is a model of Gitik and Magidor [6] in which a positive answer to Woodin's question for a large singular κ was claimed. However in his PhD thesis [16], Sharon was able to show that Gitik and Magidor's forcing adds a weak square sequence. This motivated the following theorem of Gitik and Sharon [7]:

Theorem 0.5. *It is consistent relative to a supercompact cardinal that there is a singular strong limit cardinal κ of cofinality ω where $2^\kappa = \kappa^{++}$ and \square_κ^* fails.*

In fact they show that a weaker principle, *the approachability property*, fails at κ . Cummings and Foreman [2] were able to improve this to show that there is a PCF theoretic object called a *bad scale* in Gitik and Sharon's model which implies the failure of the approachability property. In the same paper of Gitik and Sharon, the same result is obtained with $\kappa = \aleph_{\omega^2}$. It is not difficult to show that the argument of Cummings and Foreman can be carried out at \aleph_{ω^2} as well.

So this gives nice positive answer to Woodin's question for a general singular κ and even \aleph_{ω^2} , but the original question for \aleph_ω remained open. The authors' paper [19] arose as a natural attempt to push the ideas of Gitik and Sharon, and Cummings and Foreman down to \aleph_ω . In that paper we obtain a model for the failure of SCH at \aleph_ω along with the surprising result that the main forcing adds $\square_{\aleph_\omega}^*$ in a natural way. The argument that weak

square holds in our model is an abstraction of Sharon's proof that Gitik and Magidor's forcing adds weak square.

As mentioned above, in this paper we analyze scales in the extension by the main forcing from our previous paper. In particular we focus on the existence or non-existence of *very good scales* and derive the main theorem as a corollary. The paper is divided into sections as follows: In Section 1 we give some set theoretic background as well as the relevant definitions from PCF theory. In Section 2 we give the definition of the main forcing and the modified preparation forcing which we need for some of our analysis. In Section 3 we prove the bounding lemmas which provide the key tool for analysis of scales. In Sections 4 through 6 we analyze scales in different products. The main theorem is derived from work in Section 5.

1. PRELIMINARIES

The main arguments of the paper use the technique of forcing. As a standard reference for facts about forcing we recommend [8]. The most basic posets which play a role are the Cohen poset to add subsets to a regular cardinal cardinal and the Levy collapse [12].

Definition 1.1. *Let κ be a regular cardinal and $\tau > \kappa$ be an ordinal.*

- (1) $\text{Add}(\kappa, \tau)$ is the poset of less than κ sized partial functions from $\kappa \times \tau$ to 2 ordered by reverse inclusion.
- (2) $\text{Coll}(\kappa, \tau)$ is the collection of less than κ sized partial functions from κ to τ ordered by reverse inclusion.
- (3) $\text{Coll}(\kappa, < \tau)$ is the less than κ support product of $\text{Coll}(\kappa, \alpha)$ for $\kappa \leq \alpha < \tau$.

We recall a standard selection of closure and chain condition properties.

Definition 1.2. *Let κ be a regular cardinal and \mathbb{P} be a poset.*

- (1) \mathbb{P} is κ -cc if every antichain in \mathbb{P} has size less than κ .
- (2) \mathbb{P} is κ -closed if every decreasing sequence of length less than κ has a lower bound.
- (3) \mathbb{P} is κ -directed closed if every directed set of size less than κ has a lower bound.
- (4) \mathbb{P} is $< \kappa$ -distributive if the intersection of fewer than κ dense open sets is dense, equivalently if forcing with \mathbb{P} does not add any sequences of length less than κ .
- (5) For an ordinal α , \mathbb{P} is α -strategically closed if Player I has a winning strategy in the following game of length α . In this game Player I and Player II collaborate to create a decreasing sequence in \mathbb{P} where Player I plays at even ordinal stages and Player II plays at odd stages, in particular Player I plays at limit stages. Player I wins if she can always complete the construction of an α -sequence.

We will also make use of Easton's Lemma [5], which appears as Lemma 15.19 of [8] and a theorem of Laver [11].

Lemma 1.3. *Suppose that \mathbb{P} is τ -cc and \mathbb{R} is τ -closed. If G is \mathbb{P} -generic, then in $V[G]$ \mathbb{R} is $< \tau$ -distributive.*

Theorem 1.4. *Let κ be a supercompact cardinal. There is a forcing extension in which the supercompactness of κ is preserved by any further κ -directed closed forcing.*

The main type of forcing in the paper is Prikry forcing, in particular supercompact Prikry forcing. We refer the reader to Magidor's original paper [13] for a full account and repeat some of the necessary definitions here.

Definition 1.5. *Let $\kappa < \lambda$ be cardinals. We define $\mathcal{P}_\kappa(\lambda)$ to be the collection of subsets of λ of size less than κ .*

The main idea of supercompact Prikry forcing is to add a subset increasing ω sequence of elements of $\mathcal{P}_\kappa(\lambda)$ whose union is λ without collapsing κ . The forcing will clearly add a surjection from κ onto λ . To define such a sequence we need our subset increasing sequence to have a stronger property.

Definition 1.6. *Let $x, y \in \mathcal{P}_\kappa(\lambda)$. We write $x \prec y$ if $x \subseteq y$ and $|x| < \kappa \cap y$.*

Note that if $\lambda < \lambda'$, then $\mathcal{P}_\kappa(\lambda) \subseteq \mathcal{P}_\kappa(\lambda')$. So $x \prec y$ makes sense even if we have $x \in \mathcal{P}_\kappa(\lambda)$ and $y \in \mathcal{P}_\kappa(\lambda')$. Supercompactness measures allow us to control the increasing sequence which is added by initial segments. The following lemma is standard.

Lemma 1.7. *If U_n is a supercompactness measure on $\mathcal{P}_\kappa(\kappa^{+n})$, then there is a set $X_n \in U_n$ such that for all $x \in X_n$, $\kappa \cap x \in \kappa$ is an inaccessible cardinal and for all $i \leq n$, o.t. $(x \cap \kappa^{+i}) = (\kappa \cap x)^{+i}$.*

For notational convenience, we set $\kappa_x = x \cap \kappa$. The main forcing in this paper will be a *diagonal* supercompact Prikry forcing. In particular we will have countably many measures U_n on $\mathcal{P}_\kappa(\kappa^{+n})$ and we will add a \prec -increasing sequence $\langle x_n \mid n < \omega \rangle$ so that each $x_n \in \mathcal{P}_\kappa(\kappa^{+n})$ and $\bigcup_{n < \omega} x_n = \kappa^{+\omega}$.

This completes the necessary forcing background. We move on to some background from PCF theory. PCF theory was developed by Shelah [17, 18]. We refer the interested reader to [1] for an account general PCF theory and to [3] for some connections between PCF theory and combinatorics at singular cardinals. In relation to the main theorem, we illustrate the connection between PCF theory and the weak square principles from the introduction. We give the relevant definitions in the special case of a singular cardinal of cofinality ω . Let ν be such a singular cardinal and $\langle \nu_i \mid i < \omega \rangle$ be an increasing and cofinal sequence of regular cardinals less than ν . For $f, g \in \prod_{i < \omega} \nu_i$, we write $f <^* g$ if there is $j < \omega$ such that for all $i \geq j$, $f(i) < g(i)$. A sequence $\langle f_\alpha \mid \alpha < \tau \rangle$ is a *scale of length τ in $\prod_{i < \omega} \nu_i$* if it is increasing and cofinal in $\prod_{i < \omega} \nu_i$ under $<^*$. Scales are the main objects of study in PCF theory.

Theorem 1.8 (Shelah). *Every singular cardinal ν has a scale of length ν^+ in some product.*

Scales with extra properties are connected to weak square principles.

Definition 1.9. *Let $\langle f_\alpha \mid \alpha < \tau \rangle$ be a scale in some product $\prod_{i < \omega} \nu_i$. An ordinal γ with $\text{cf}(\gamma) > \omega$ is a good point (resp. very good point) for \vec{f} if there are $A \subseteq \gamma$ unbounded (resp. club) and $j < \omega$ such that for all $i \geq j$, $\langle f_\alpha(i) \mid \alpha \in A \rangle$ is strictly increasing.*

Definition 1.10. *A scale of length τ in some product is good (resp. very good) if modulo a club every point of uncountable cofinality is good (resp. very good). Further a bad scale is one that is not good.*

The following two theorems show the connection between goodness of scales and weak square principles.

Theorem 1.11 (Shelah). *If \square_ν^* holds then every scale of length ν^+ is good.*

Theorem 1.12 (Cummings-Foreman-Magidor [3]). *If $\square_{\nu,\lambda}$ holds for some $\lambda < \nu$, then every product which carries a scale of length ν^+ also carries a very good scale of length ν^+ .*

From Shelah's theorem above, we see that in the extension by the main forcing from our previous paper all scales of length $\aleph_{\omega+1}$ are good. The theorem of Cummings, Foreman and Magidor motivates our analysis of very good scales. In particular we will show that some products can be forced to have no very good scale. The main theorem is then a corollary of the non-existence of very good scales and the above theorem.

2. THE MAIN FORCING

Let V_0 be a model where κ is indestructibly supercompact and GCH holds above κ . For clarity below we set $\mu = \kappa^{+\omega+1}$. We use the same preparation as Lambie-Hanson [10]. Let \mathbb{A} be the reverse Easton support iteration of $\prod_{n < \omega} \text{Add}(\alpha^{+n}, \alpha^{+\omega+2})$ for $\alpha \leq \kappa$ inaccessible, let E be \mathbb{A} -generic and let $V = V_0[E]$. It is well-known that iterations like \mathbb{A} preserve the supercompactness of κ , but we will need to be more careful. In particular we wish to have a measure in V which extends a similar measure in V_0 with some special properties. For this we will need Lemma 3.1 of Lambie-Hanson [10] which is a modification of an argument of Gitik and Sharon [7]. Note that we avoid any interaction between the Laver preparation used to obtain V_0 and the iteration \mathbb{A} .

Lemma 2.1. *Let \bar{U} be a supercompactness measure in V_0 on $\mathcal{P}_\kappa(\mu)$ whose projection to a measure on κ concentrates on α where the Laver preparation selects the trivial forcing. In V there is a supercompactness measure U on $(\mathcal{P}_\kappa(\mu))_V$ such that*

$$(1) \ U \supseteq \bar{U},$$

- (2) The ultrapower map $j : V_0[E] \rightarrow M_0[F]$ generated by U is a lift of the ultrapower map generated over V_0 by \bar{U} for some $j(\mathbb{A})$ -generic object F over M_0 and
- (3) for each $n < \omega$ and each $\beta < j(\kappa^{+n})$ there is a function $g_\beta^n : \kappa^{+n} \rightarrow \kappa^{+n}$ such that $j(g_\beta^n)(\text{sup } j \text{ ``}\kappa^{+n}\text{'}) = \beta$.

Note that when $n = 0$ condition (3) of the lemma is Lemma 2.26 in Gitik and Sharon's paper [7]. The $n = 0$ case is also used in the authors' first paper as it ensures that certain factor maps have high critical points.

We work in V to define a variant of the main forcing from our previous paper. The idea is to define a diagonal supercompact Prikry forcing with interleaved collapses. The main observation from our previous paper is that the collapses between the Prikry points must be taken from V_0 . To define the forcing we will need supercompactness measures U_n on $\mathcal{P}_\kappa(\kappa^{+n})$ which are projections of U and sets $X_n \in U_n$ as in Lemma 1.7 with $X_n \subseteq (\mathcal{P}_\kappa(\kappa^{+n}))_{V_0}$.

Definition 2.2. Conditions in \mathbb{P} are of the form $p = \langle d_0, d_1, \langle p_n \mid n < \omega \rangle \rangle$, where setting $l = \text{lh}(p)$, we have:

- (1) For $0 \leq n < l$, $p_n = \langle x_n, c_n \rangle$ such that:
- $x_n \in X_n$, and for $i < n$, $x_i \prec x_n$,
 - $c_0 \in \text{Coll}(\kappa_{x_0}, < \kappa_{x_1})_{V_0}$ if $1 < l$, and if $l = 1$, $c_0 \in \text{Coll}(\kappa_{x_0}, < \kappa)_{V_0}$.
 - if $1 < l$, for $0 < n < l - 1$, $c_n \in \text{Coll}(\kappa_{x_n}^{+n+2}, \kappa_{x_n}^{+\omega+1})_{V_0} \times \text{Coll}(\kappa_{x_n}^{+\omega+2}, < \kappa_{x_{n+1}})_{V_0}$, and $c_{l-1} \in \text{Coll}(\kappa_{x_{l-1}}^{+l+1}, \kappa_{x_{l-1}}^{+\omega+1})_{V_0} \times \text{Coll}(\kappa_{x_{l-1}}^{+\omega+2}, < \kappa)_{V_0}$.
- (2) For $n \geq l$, $p_n = \langle A_n, C_n \rangle$ such that:
- $A_n \in U_n$, $A_n \subseteq X_n$, and $x_{l-1} \prec y$ for all $y \in A_n$,
 - C_n is a function with domain A_n , for $y \in A_n$, $C_n(y) \in \text{Coll}(\kappa_y^{+n+2}, \kappa_y^{+\omega+1})_{V_0} \times \text{Coll}(\kappa_y^{+\omega+2}, < \kappa)_{V_0}$ if $n > 0$, and $C_n(y) \in \text{Coll}(\kappa_y, < \kappa)_{V_0}$ if $n = 0$,
- (3) if $l > 0$, then there is $\delta < \kappa_{x_0}$ (which we call δ^p) such that $(d_0, d_1) \in \text{Coll}(\omega, \delta^+)_{V_0} \times \text{Coll}(\delta^{+++}, < \kappa_{x_0})_{V_0}$, otherwise $d_0 \in \text{Coll}(\omega, \kappa)_{V_0}$ and $d_1 = \emptyset$.

$q = \langle d_0^q, d_1^q, \langle q_n \mid n < \omega \rangle \rangle \leq p = \langle d_0^p, d_1^p, \langle p_n \mid n < \omega \rangle \rangle$ if $\text{lh}(q) \geq \text{lh}(p)$ and:

- if $\text{lh}(p) > 0$, then $\delta^p = \delta^q$ and $(d_0^q, d_1^q) \leq (d_0^p, d_1^p)$,
- if $\text{lh}(p) = 0$, then $d_0^q \leq d_0^p$,
- for all $n < \text{lh}(p)$, $x_n^q = x_n^p, c_n^q \leq c_n^p$,
- for $\text{lh}(p) \leq n < \text{lh}(q)$, $x_n^q \in A_n^p$ and $c_n^q \leq C_n^p(x_n^q)$,
- for $n \geq \text{lh}(q)$, $A_n^q \subseteq A_n^p$ and for all $y \in A_n^q$, $C_n^q(y) \leq C_n^p(y)$

$q \leq^* p$, i.e. q is a direct extension of p , if $q \leq p$ and $\text{lh}(q) = \text{lh}(p)$.

Definition 2.3. For $p \in \mathbb{P}$, we let $s(p) = \langle d_0^p, d_1^p, \langle p_n \mid n < \text{lh}(p) \rangle \rangle$, which we call the stem of p .

The key difference between this version of the forcing and the version from our previous paper is the selection of ω_1 . The poset \mathbb{P} above selects a

cardinal δ and forces $\delta^{++} = \omega_1$ in the extension. In the argument below we make a careful selection of δ in order to ensure that there are no very good scales in some product in the extension.

We must work a little to see that \mathbb{P} has the same properties as in our previous paper. We note that the selection mechanism for ω_1 cannot interfere with the proofs of properties of the poset, since we can always extend to a condition of length at least one at which point the poset is nearly identical to the poset \mathbb{P} of [19]. The Prikry Lemma, the preservation of μ and μ^+ , the fact that $\kappa = \aleph_\omega$ in the extension and the characterization of genericity are abstract enough that they only rely on the distributivity of the collapses. So it is enough to show the following lemma, which shows that different preparation did not effect the distributivity of the collapses.

Lemma 2.4. *There is a U_0 -measure one set of α such that for all $n < \omega$ and all λ , the poset $\text{Coll}(\alpha^{+n+2}, \alpha^{+\omega+1})_{V_0} \times \text{Coll}(\alpha^{+\omega+2}, \lambda)_{V_0}$ is $< \alpha^{+n+2}$ -distributive in V .*

Proof. Work in V_0 . Let \mathbb{A}^α be $\mathbb{A} \upharpoonright \alpha+1$ -term forcing for a tail of the iteration. Let \mathbb{B} be the $\mathbb{A} \upharpoonright \alpha$ -term forcing for $\prod_{m \geq n+2} \text{Add}(\alpha^{+m}, \alpha^{+\omega+2})$. It follows that both \mathbb{A}^α and \mathbb{B} are α^{+n+2} -closed. Moreover $\mathbb{A} \upharpoonright \alpha * \prod_{m < n+2} \text{Add}(\alpha^{+m}, \alpha^{+\omega+2})$ is α^{+n+2} -cc.

The product $\mathbb{A}^\alpha \times \mathbb{B} \times \mathbb{A} \upharpoonright \alpha * \prod_{m < n+2} \text{Add}(\alpha^{+m}, \alpha^{+\omega+2})$ generates a generic object for \mathbb{A} . Hence by Lemma 1.3 any $< \alpha^{+n+2}$ -sequence from an extension by

$$\mathbb{A} \times \text{Coll}(\alpha^{+n+2}, \alpha^{+\omega+1}) \times \text{Coll}(\alpha^{+\omega+2}, \lambda)$$

is in the inner model determined by $\mathbb{A} \upharpoonright \alpha * \prod_{m < n+2} \text{Add}(\alpha^{+m}, \alpha^{+\omega+2})$. \square

Remark 2.5. A similar argument establishes the conclusion of the previous lemma with κ in place of α .

We also recall that there are posets $\mathbb{C}_n = (\text{Coll}(\kappa^{+n+2}, \kappa^{+\omega+1}) \times \text{Coll}(\kappa^{+\omega+2}, < j(\kappa)))_{V_0}$ and $\mathbb{D} = \prod \mathbb{C}_n / \text{fin}$. From our previous paper, Proposition 3.13 and Lemma 3.14 show that \mathbb{P} projects to \mathbb{D} in a natural way and the quotient forcing \mathbb{P}/\mathbb{D} is μ -cc. Clearly \mathbb{D} is a poset in V_0 and by work in Section 4 of our previous paper it adds a $\square_{\kappa^+}^*$ -sequence.

As a final piece of preparation, we restate Lemma 3.8 from our previous paper because it will be used often below.

Lemma 2.6. *Let D be a dense open subset of \mathbb{P} and $p \in \mathbb{P}$. There is $p_0 \leq^* p$ such that $s(p) = s(p_0)$ and for all $q \leq p_0$, if $q \in D$, then $s(q) \cap p_0 \upharpoonright [\text{lh}(q), \omega) \in D$.*

3. BOUNDING LEMMAS

In this section we prove bounding lemmas for all relevant products in the final generic extension. The bounding lemmas will be used to analyze scales in these different products.

Let $\langle x_n \mid n < \omega \rangle$ be the Prikry sequence added by \mathbb{P} , each $x_n \in \mathcal{P}_\kappa(\kappa^{+n})$, and let $\lambda_n := \kappa_{x_n}$. Following Lambie-Hanson we prove a bounding lemma which will help us analyze scales in $\prod_{\substack{i \leq n \\ n < \omega}} \lambda_{n+1}^{+i}$.

Lemma 3.1. *Let $p \in \mathbb{P}$ and let \dot{h} be a \mathbb{P} -name for an element of the product*

$$\prod_{\substack{n < \omega \\ i \leq n}} x_{n+1} \cap \kappa^{+i}$$

then there are $q \leq^ p$ and $\langle H_{n,i} \mid n < \omega, i \leq n \rangle$ with $H_{n,i} : \mathcal{P}_\kappa(\kappa^{+n}) \rightarrow \kappa^{+i}$ such that q forces for all large n and all $i \leq n$, $\dot{h}(n,i) \leq H_{n,i}(\dot{x}_n)$.*

Proof. Using the Prikry Lemma and an easy diagonal construction, we may assume that for all $q \leq p$ of length $n+2$, if $q \parallel \text{“}\dot{h}(n,i) = \check{\gamma}\text{”}$, then $s(q) \frown p \upharpoonright [n+2, \omega) \parallel \text{“}\dot{h}(n,i) = \check{\gamma}\text{”}$. Let s be a lower part of length $n+1$ and $x \in \mathcal{P}_\kappa(\kappa^{+n+1})$ so that $s \frown x$ is a lower part. Note that any condition with lower part extending $s \frown x$ forces that $\dot{h}(n,i) \in x \cap \kappa^{+i}$ for all $i \leq n$.

Let $i \leq n$ and $\gamma \in x \cap \kappa^{+i}$. Let $E_{s,\gamma,i}$ be the set of all $c \in \mathbb{C}_x := \text{Coll}(\kappa_x^{+n+2}, \kappa_x^{+\omega+1})_{V_0} \times \text{Coll}(\kappa_x^{+\omega+2}, < \kappa)_{V_0}$, such that there is $s' \leq s$ such that $s' \frown \langle x, c \rangle \frown p \upharpoonright [n+2, \omega) \parallel \dot{h}(n,i) = \check{\gamma}$. We claim that $E_{s,\gamma,i}$ is dense open in \mathbb{C}_x below $C_n^p(x)$. This follows from an application of the Prikry Lemma to a condition of the form $s \frown \langle x, c \rangle \frown p \upharpoonright [n+2, \omega)$ and the choice of p . \mathbb{C}_x is $< \kappa_x^{+n+2}$ -distributive and hence

$$E_x = \bigcap_{\substack{s \in S_x \\ \gamma \in x \\ i \leq n}} E_{s,\gamma,i}$$

is dense. We find a direct extension p' of p such that for all x , $C_{n+1}^{p'}(x) \in E_x$ where n is such that $x \in \mathcal{P}_\kappa(\kappa^{+n+1})$.

Now we claim that for each stem s of length $n+1$ compatible with p' , each $i \leq n$ and each $x \in A_{n+1}^p$, there is $s' \leq s$ such that $s' \frown \langle x, C_{n+1}^{p'}(x) \rangle \frown p \upharpoonright [n+2, \omega)$ decides $\dot{h}(n,i)$. Let $r \leq s \frown p' \upharpoonright [n+1, \omega)$ of length at least $n+2$ decide $\dot{h}(n,i)$ with value γ . Now by construction $C_{n+1}^{p'}(x_{n+1}^r) \in E_{s(r)|_{n+1}, \gamma, i}$. Any s' witnessing this fact is as required, since $s' \frown \langle x_{n+1}^r, C_{n+1}^{p'}(x_{n+1}^r) \rangle \frown p \upharpoonright [n+2, \omega)$ decides $\dot{h}(n,i) = \check{\gamma}$ and is compatible with r .

Let \vec{y} be the Prikry part of a stem s of length $n+1$. Let $\mathbb{C}_{\vec{y}}$ be the product of Levy Collapses from which the collapse conditions in s are taken. By the previous paragraph for each \vec{y} of length $n+1$, each $i \leq n$ and each x above \vec{y} , there is a maximal antichain $F_{\vec{y},x}^i \subseteq \mathbb{C}_{\vec{y}} \upharpoonright \kappa_x$ such that for all $\vec{c} \in F_{\vec{y},x}^i$, the condition $\vec{y} + \vec{c} \frown \langle x, C_{n+1}^{p'}(x) \rangle \frown p \upharpoonright [n+2, \omega)$ decides the value of $\dot{h}(n,i)$.

A few notes are in order. $\vec{y} + \vec{c}$ is the natural stem obtained by interleaving the elements of \vec{y} and \vec{c} . The antichain is only maximal below the weakest condition in $\mathbb{C}_{\vec{y}}$ which is compatible with information from p' (both

constraint information and collapses in $s(p')$). The antichain $F_{\vec{y},x}^i$ has size less than κ_x , since it is contained in a κ_x -cc poset.

Let $\alpha_{\vec{y},x}^i$ be the least element of x greater than all values γ witnessing that some $\vec{c} \in F_{\vec{y},x}^i$. Clearly this supremum is less than $\sup(x \cap \kappa^{+i})$, since it has size less than κ_x . Now for a fixed i and \vec{y} as above the map $x \mapsto \alpha_{\vec{y},x}^i$ is regressive and hence constant on a measure one set $A_{\vec{y}}^i$ with value $\alpha_{\vec{y}}^i$. Let $A_{\vec{y}} = \bigcap_{i \leq n} A_{\vec{y}}^i$ and let $\langle A_n^* \mid n < \omega \rangle$ be the sequence of measure one sets obtained from diagonally intersecting the $A_{\vec{y}}$ for \vec{y} a Prikry stem. Let q be the restriction of p' to $\langle A_n^* \mid n < \omega \rangle$ and define $H_{n,i}(y)$ to be the supremum of $\alpha_{\vec{y}}^i$ such that \vec{y} ends with y .

We claim that q and $H_{n,i}$ are as required. Let $r \leq q$ of length at least $n+2$ decide the value of $\dot{h}(n,i)$ to be γ . Let \vec{y} be the Prikry part of $s(r)$ of length $n+1$ and let $x = x_{n+1}^r$. It follows that the collapse part of the stem of r restricted to $n+1$ is compatible with some $\vec{c} \in F_{\vec{y},x}^i$. Moreover the value of $\dot{h}(n,i)$ that witnesses $\vec{c} \in F_{\vec{y},x}^i$ is γ , since r is compatible with $\vec{y} + \vec{c} \frown \langle x, C_{n+1}^{p'}(x) \rangle \frown p \upharpoonright [n+2, \omega]$. So $\gamma < \alpha_{\vec{y},x}^i$ and $\alpha_{\vec{y},x}^i = \alpha_{\vec{y}}^i$ since $x \in A_{\vec{y}}$. Now by the definition of $H_{n,i}$, $\gamma < H_{n,i}(y)$ where y is the top point of \vec{y} . \square

We now work towards bounding lemmas for the products $\prod \lambda_n^{+n}$ and $\prod \lambda_n^{+n+1}$. We start with some notation. Recall that we denoted $\mathbb{C}_x := \text{Coll}(\kappa_x^{+n+2}, \kappa_x^{+\omega+1})_{V_0} \times \text{Coll}(\kappa_x^{+\omega+2}, < \kappa)_{V_0}$.

Proposition 3.3 from our previous paper shows that $(\text{Coll}(\kappa^{+n+2}, \kappa^{+\omega+1}) \times \text{Coll}(\kappa^{+\omega+2}, < j(\kappa)))_{\text{Ult}(V_0, \bar{U})} \simeq [x \mapsto \mathbb{C}_x]_{U_n}$. By closure of the ultrapower, we also have that $(\text{Coll}(\kappa^{+n+2}, \kappa^{+\omega+1}) \times \text{Coll}(\kappa^{+\omega+2}, < j(\kappa)))_{\text{Ult}(V_0, \bar{U})}$ is the same as \mathbb{C}_n . So, $\mathbb{C}_n \simeq [x \mapsto \mathbb{C}_x]_{U_n}$. So each \mathbb{C}_n can be viewed as a poset in V_0 . Recall that $\mathbb{D} = \prod_n \mathbb{C}_n / \text{finite}$. We set $\mathbb{D}^* = \prod_n \mathbb{C}_n$ and for each n , $\mathbb{D}_n = \prod_{k \geq n} \mathbb{C}_k$. Then working in V_0 , \mathbb{D}^* projects to \mathbb{D}_n , which in turn projects to \mathbb{D} , each \mathbb{D}_n is κ^{+n+2} -closed, and $\mathbb{D}^* / \mathbb{D}_n$ is κ^{++} -closed. So in particular it can be shown that \mathbb{D} is $< \kappa^{+\omega}$ distributive over V .

Let H be \mathbb{D} -generic over V .

Lemma 3.2. *Suppose that $1 \Vdash_{\mathbb{P}/H} \dot{f} \in \prod_n \dot{\lambda}_n^{+n+1}$. Then there is a sequence of functions $\langle H_n \mid n < \omega \rangle$ in $V[H]$, such that $\text{dom}(H_n) = \mathcal{P}_\kappa(\kappa^{+n})$, $H_n(x) < \kappa_x^{+n+1}$ for all x , and $1 \Vdash_{\mathbb{P}/H} \dot{f}(n) < H_n(\dot{x}_n)$.*

Proof. By Lemma 2.6, we can find a condition p with length 1, such that for all n , if there is q , such that $q \parallel \dot{f}(n)$ and $q \upharpoonright [\text{lh}(q), \omega] \leq p \upharpoonright [\text{lh}(q), \omega]$, then $s(q) \frown p \upharpoonright [\text{lh}(q), \omega] \parallel \dot{f}(n)$. Also, note that if q has length at least $n+1$, then $q \Vdash \dot{f}(n) < \kappa_x^{+n+1}$, where $x = x_n^q$.

Fix $0 < n < \omega$ and $x \in \mathcal{P}_\kappa(\kappa^{+n})$. For $c \in \mathbb{C}_x$, define $H_n(x, c) = \sup\{\gamma < \kappa_x^{+n+1} \mid (\exists q) \text{lh}(q) = n+1, q_n = \langle x, c \rangle, q \upharpoonright [n+1, \omega] \leq p \upharpoonright [n+1, \omega] \text{ and } q \Vdash \dot{f}(n) = \gamma\}$. Let S_x be the set of all stems s of length n , such that $s \frown \langle x, 1_{\mathbb{C}_x} \rangle$ is a stem. Since $|S_x| < \kappa_x^{+n}$, and by the way we prepared p , we get that $H_n(x, c) < \kappa_x^{+n+1}$.

Claim 3.3. For $s \in S_x$ and $\gamma < \kappa_x^{+n+1}$, the set $D_{s,\gamma} := \{\langle c, b \rangle \in \mathbb{C}_x \times \mathbb{D}_{n+1} \mid (\exists q)(\text{lh}(q) = n+1, q \upharpoonright n \leq s, q_n = \langle x, c \rangle, [\vec{C}^q] = b, q \parallel \text{“}\dot{f}(n) = \gamma\text{”})\}$ is dense.

By distributivity of $\mathbb{C}_x \times \mathbb{D}_{n+1}$, $D_x := \bigcap_{s \in S_x, \gamma < \kappa_x^{+n+1}} D_{s,\gamma}$ is also dense.

Claim 3.4. For $s \in S_x$, the set $E_s := \{c \in \mathbb{C}_x \mid \exists s' \leq s, s' \frown \langle x, c \rangle \frown p \upharpoonright [n+1, \omega] \Vdash \text{“}\dot{f}(n) < H_n(x, c)\text{”}\}$ is open dense.

Proof. Clearly E_s is open. We prove its density. Let $s \in S_x, c \in \mathbb{C}_x$. Let $\langle c', b \rangle \in D_x$ be below $\langle c, [\vec{C}^p \upharpoonright [n+1, \omega]] \rangle$. Let $q \leq s \frown \langle x, c' \rangle \frown b$ be such that $q \Vdash \dot{f}(n) = \gamma$ for some $\gamma < \kappa_x^{+n+1}$. Since $\langle c', b \rangle \in D_{q \upharpoonright n, \gamma}$, there is some $r \parallel \dot{f}(n) = \gamma$, such that $r \upharpoonright n+1 = s' \frown \langle x, c' \rangle$, $[\vec{C}^r] = b$ and $s' \leq q \upharpoonright n$. But then q and r are compatible, so $r \Vdash \dot{f}(n) = \gamma$. By the way we prepared p , it follows that $s(r) \frown p \upharpoonright [n+1, \omega] \Vdash \dot{f}(n) = \gamma$. Then $s(r) \frown p \upharpoonright [n+1, \omega] \Vdash \text{“}\dot{f}(n) < H_n(x, c')\text{”}$, and so $c' \in E_s$, witnessed by s' . \square

Using distributivity of \mathbb{C}_x , we get that $E_x^n := \bigcap_{s \in S_x} E_s$ is open dense.

Set $E_n := [x \mapsto E_x^n]_{U_n}$. Then there is $\langle b_n \mid n < \omega \rangle$, such that $\langle b_n \mid n < \omega \rangle / \text{finite} \in H$ and each $b_n \in E_n$. Denote $b_n = [x \mapsto b_n(x)]_{U_n}$, and we may assume that for every x , $b_n(x) \in E_x^n$. In $V[H]$, define $H_n(x) := H_n(x, b_n(x))$.

We claim that $1 \Vdash_{\mathbb{P}/H} \text{“}\exists \bar{n} \forall n \geq \bar{n}, \dot{f}(n) < H_n(\dot{x}_n)\text{”}$. Let G be \mathbb{P}/H -generic. For each n , let \mathcal{C}_n be the \mathbb{C}_{x_n} -generic, induced by G . By genericity of the Prikry sequence, for some \bar{n} , for all $n \geq \bar{n}$:

- (1) $c_n := b_n(x_n) \in E_{x_n}^n \cap \mathcal{C}_n$,
- (2) $x_n \in A_n^p$, and
- (3) $C_n^p(x_n) \in \mathcal{C}_n$.

Now let $n \geq \bar{n}$ and $q \in G$ be such that $\text{lh}(q) \geq n+1$ and $q \Vdash \dot{f}(n) = \gamma$ for some γ . By items (2) and (3) above, we have that $q \upharpoonright [\bar{n}+1, \omega]$ and $p \upharpoonright [\bar{n}+1, \omega]$ are compatible. (We can even arrange that $q \upharpoonright [\bar{n}+1, \omega] \leq p \upharpoonright [\bar{n}+1, \omega]$.) Then by the way we prepared p , we have that $s(q) \frown p \upharpoonright [\text{lh}(q), \omega] \Vdash \dot{f}(n) = \gamma$. Also, we may assume that $c_n^q \leq c_n$. Let $s = q \upharpoonright n$. Since $c_n \in E_{x_n}^n$, there is some $s' \leq s$, such that

$$q' := s' \frown \langle x_n, c_n \rangle \frown p \upharpoonright [n+1, \omega] \Vdash \text{“}\dot{f}(n) < H_n(\dot{x}_n, c_n)\text{”}.$$

But q' and $s(q) \frown p \upharpoonright [\text{lh}(q), \omega]$ are compatible, so $\gamma < H_n(x_n, c_n) = H_n(x_n)$. \square

Remark 3.5. Note that in the final argument we could have assumed that items (2), (3) hold only for all $n > \bar{n}$.

Remark 3.6. Let $p, \langle b_n \mid n < \omega \rangle, \langle H_n \mid n < \omega \rangle$ be as in the proof above. If q, \bar{n} are such that:

- for all $n > \bar{n}$,
 - if $n < \text{lh}(q)$, $c_n^q \leq C_n^p(x_n^q)$, and
 - if $n \geq \text{lh}(q)$, $A_n^q \subset A_n^p$ and for all $x \in A_n^q$, $C_n^q(x) \leq C_n^p(x)$,

- for all $n \geq \bar{n}$,
 - if $n < \text{lh}(q)$, $c_n^q \leq b_n(x_n^q)$, and
 - if $n \geq \text{lh}(q)$, for all $x \in A_n^q$, $C_n^q(x) \leq b_n(x)$,

then $q \Vdash_{\mathbb{P}/H} “(\forall n \geq \bar{n}) \dot{f}(n) < H_n(\dot{x}_n)”$.

Remark 3.7. Arguing as above, we get the same bounding lemma for functions in $\prod_n \lambda_n^{+n}$. Slightly more complicated, but essentially the same arguments as above, work for functions in $\prod_n \lambda_n^{+n+3}$. The difference here is that we define $H_n(x, c)$ for $c \in \text{Coll}(\kappa_x^{+\omega+2}, < \kappa)_V$, and consider the dense sets $E_{s \smallfrown d}$ and $D_{s \smallfrown d, \gamma}$, for all $s \in S_x$ and $d \in \text{Coll}(\kappa_x^{+n+2}, \kappa_x^{+\omega+1})_V$.

The remaining product is of the form $\prod \lambda_n^{+n+2}$. As in [7] fix functions $\langle F_\beta \mid \beta < \mu \rangle$ in V , such that each $F_\beta : \kappa \rightarrow \kappa$ and for all n , $j_U(F_\beta)(\kappa) = \beta$. Here U is the top measure on $\mathcal{P}_\kappa(\mu)$.

Let H^* be \mathbb{D}^*/H -generic. And for each n , let K_n be the generic on \mathbb{C}_n , induced from H^* .

Lemma 3.8. *Suppose that in $V[G]$, $f \in \prod_n \lambda_n^{+n+2}$. Then working in $V[H^*]$, we can define $\langle \gamma_n \mid n < \omega \rangle$, such that each $\gamma_n < (\kappa^{+n+2})^V$ and for all large n , $f(n) < F_{\gamma_n}(\lambda_n)$.*

Proof. Suppose that $1 \Vdash \dot{f} \in \prod_n \dot{\lambda}_n^{+n+2}$. Arguing as in the proof of the Prikry lemma, we can find a condition p with length 1, such that for all $x \in \mathcal{P}_\kappa(\kappa^{+n})$, for all $\gamma < \kappa_x^{+n+2}$, for all stems h of length $n+1$ ending in x , there is a stem $h' \leq h$, such that $h' \smallfrown p \upharpoonright [n+1, \omega) \parallel \dot{f}(n) = \gamma$. To do that, we use that $D_{h, \gamma} := \{b \in \mathbb{D}_{n+1} \mid (\exists q)(\text{lh}(q) = n+1, q \upharpoonright n+1 \leq h, [\vec{C}^q] = b, q \parallel “\dot{f}(n) = \gamma”)\}$ is dense. Then $D^n := \bigcap_{h, \gamma} D_{h, \gamma}$ is also dense, since the number of possible stems h of length $n+1$ is κ^{+n} , and \mathbb{D}_{n+1} is $< \kappa^{+n+3}$ -distributive over V .

Fix $0 < n < \omega$ and $x \in \mathcal{P}_\kappa(\kappa^{+n})$. For $c \in \mathbb{C}_x$, define $H_n(x, c) = \sup\{\gamma < \kappa_x^{+n+2} \mid (\exists q) \text{lh}(q) = n+1, q_n = \langle x, c \rangle, q \upharpoonright [n+1, \omega) \leq p \upharpoonright [n+1, \omega) \text{ and } q \Vdash \dot{f}(n) = \gamma\}$. Let S_x be the set of all stems s of length n , such that $s \smallfrown \langle x, 1_{\mathbb{C}_x} \rangle$ is a stem. Since $|S_x| < \kappa_x^{+n}$, and by the way we prepared p , we get that $H_n(x, c) < \kappa_x^{+n+2}$.

Set $\gamma_n = \sup_{b \in K_n} [x \mapsto H_n(x, b(x))]$. Here for $b \in K_n$, we denote $b = [x \mapsto b(x)]_{U_n}$. Note that by countable closure of \mathbb{D}^* , $\langle \gamma_n \mid n < \omega \rangle$ is in V .

Claim 3.9. $\gamma_n < (\kappa^{+n+2})^V$

Proof. Let $S^* := \{[x \mapsto s_x]_{U_n} \mid (\forall x)(s_x \in S_x)\}$. Since for all x , $|S_x| < \kappa_x^{+n}$, we have that $|S^*| < \kappa^{+n}$. We claim that γ_n can be written as a supremum over S^* .

For each $s_x \in S_x$, define $H_n(x, s_x, c)$ as follows. If there is some q , such that $q \upharpoonright n+1 = s_x \smallfrown \langle x, c \rangle, q \upharpoonright [n+1, \omega) \leq p \upharpoonright [n+1, \omega)$ and $q \Vdash \dot{f}(n) = \gamma$, let $H_n(x, s_x, c) = \gamma$. Otherwise, $H_n(x, s_x, c) = 0$. Note that $H_n(x, c) = \sup_{s_x \in S_x} H_n(x, s_x, c)$.

Now, if $b, d \in K_n$, then for almost all x , $b(x)$ and $d(x)$ are compatible, and K_n preserves cardinals up to κ^{+n+2} . So for $s \in S^*$, either $[x \mapsto$

$H_n(x, s(x), b(x)) = [x \mapsto H_n(x, s(x), d(x))]$, or one of them is 0. For such $s \in S^*$, let $\gamma_n^s = \sup_{b \in K_n} [x \mapsto H_n(x, s(x), b(x))]$. By the above it follows that this is below κ^{+n+2} .

Finally, $\gamma_n = \sup_{s \in S^*} \gamma_n^s$. \square

Let $\vec{d} \in H^*$ be such that for all n , $d(n) \Vdash_{\mathbb{C}_n} \gamma_n = \sup_{b \in \dot{K}_n} [x \mapsto H_n(x, b(x))]_{U_n}$. Here \dot{K}_n is the canonical name for the generic filter; and denote $\dot{K}_n = [x \mapsto \dot{C}_n^q]_{U_n}$. Let $q \leq p$, be such that for all large n , $[C_n^q]_{U_n} \leq_{\mathbb{C}_n} d(n)$. We claim that q forces that $\langle \gamma_n \mid n < \omega \rangle$ is as desired.

Suppose G is \mathbb{P}/H -generic with $q \in G$. Then for all large n , $\{x \mid C_n^q(x) \Vdash_{\mathbb{C}_x} F_{\gamma_n}(\kappa_x) = \sup_{c \in \dot{C}_x} H_n(x, c)\} \in U_n$. In $V[G]$, let \mathcal{C}_n be the generic on \mathbb{C}_{x_n} . By genericity and since $q \in G$, for all large n , $F_{\gamma_n}(\lambda_n) = \sup_{c \in \mathcal{C}_n} H_n(x_n, c)$.

To finish the proof it is enough to prove the following claim.

Claim 3.10. *For all large n , $f(n) < \sup_{c \in \mathcal{C}_n} H_n(x_n, c)$.*

Proof. Let $r \in G$ be with length at least $n+1$, such that $r \Vdash \dot{f}(n) = \gamma$. We claim that $D := \{c \in \mathbb{C}_{x_n} \mid (\exists q) \text{lh}(q) = n+1, q_n = \langle x_n, c \rangle, q \upharpoonright [n+1, \omega] \leq p \upharpoonright [n+1, \omega] \text{ and } q \Vdash \dot{f}(n) = \gamma\}$ is dense below c_n^r . For any $c \leq c_n^r$, let $h = r \upharpoonright n \cap \langle x_n, c \rangle$. By the way we prepared p , there is $h' \leq h$, such that $r' := h' \widehat{\cap} p \upharpoonright [n+1, \omega] \Vdash \dot{f}(n) = \gamma$. But since r and r' are compatible, we have that $r' \Vdash \dot{f}(n) = \gamma$. So, $c_n^{r'} \in D$.

Now let $c \in D \cap \mathcal{C}_n$. Then $f(n) = \gamma < H_n(x_n, c)$. \square

\square

This completes our analysis of bounding lemmas. In the following sections we analyze scales from different products in the extension using the bounding lemmas above.

4. A VERY GOOD SCALE IN $\prod_{i \leq n, n < \omega} \lambda_{n+1}^{+i}$

In this section we prove the following theorem.

Theorem 4.1. *In $V[G]$ there is a very good scale of length μ^+ in the product $\prod_{\substack{i \leq n \\ n < \omega}} \lambda_{n+1}^{+i}$.*

Recall that $j : V \rightarrow M$ is a μ -supercompactness embedding and that we have for each $n < \omega$ and each $\beta < j(\kappa^{+n})$ a function $g_\beta^n : \kappa^{+n} \rightarrow \kappa^{+n}$ such that $j(g_\beta^n)(\sup j \text{``}\kappa^{+n}) = \beta$. Following Lambie-Hanson [10], we fix sequences $\langle \alpha_\zeta^i \mid \zeta < \mu^+ \rangle$ increasing and cofinal in $j(\kappa^{+i})$. We define a scale as follows. For $\zeta < \mu^+$, $n < \omega$ and $i \leq n$, we set $f_\zeta(n, i) = g_{\alpha_\zeta^i}^i(\sup(x_n \cap \kappa^{+i}))$. We will see that this scale generates a scale in the desired product. Many of the arguments are straightforward so we only sketch the proof and refer the reader to [10] for the details. The first claim identifies a product which is relevant for the f_ζ .

Claim 4.2. *For all $\zeta < \mu^+$, for all large n and for all $i \leq n$, $f_\zeta(n, i) < \sup(x_{n+1} \cap \kappa^{+i})$.*

Proof. Let $\zeta < \mu^+$, $n < \omega$ and $i \leq n$. It is not hard to see that the set $\{x \in \mathcal{P}_\kappa(\kappa^{+n+1}) \mid \text{for all } y \in \mathcal{P}_\kappa(\kappa^{+n}) \text{ with } y \prec x, g_{\alpha_\zeta}^i(\sup(y \cap \kappa^{+i})) < \sup(x \cap \kappa^{+i})\}$ is in U_{n+1} . The conclusion follows since for all large n , x_{n+1} comes from this set. \square

So we can view $\langle f_\zeta \mid \zeta < \mu^+ \rangle$ as sequence from $\prod_{\substack{i \leq n \\ n < \omega}} \sup(x_{n+1} \cap \kappa^{+i})$.

Claim 4.3. *For all $\zeta < \zeta' < \mu^+$, $f_\zeta <^* f_{\zeta'}$, ie for all large n and all $i \leq n$, $f_\zeta(n, i) < f_{\zeta'}(n, i)$.*

The proof goes by obtaining a sequence of measure one sets $B_n^i \in U_n$ such that for all $x \in B_n^i$ putting x on the Prikry sequence ensures that $f_\zeta(n, i) < f_{\zeta'}(n, i)$. Such measure one sets exist using that for all n and all $i \leq n$, $\alpha_\zeta^i < \alpha_{\zeta'}^i$, the choice of the g_β^i and the fact that $j^{\aleph_n} \cap j(\kappa^{+i}) = j^{\aleph_n}$.

Claim 4.4. *$\langle f_\zeta \mid \zeta < \mu^+ \rangle$ is cofinal in $\prod_{\substack{i \leq n \\ n < \omega}} \sup(x_{n+1} \cap \kappa^{+i})$.*

Proof. Let $h \in \prod_{\substack{i \leq n \\ n < \omega}} \sup(x_{n+1} \cap \kappa^{+i})$ in $V[G]$. We use Lemma 3.1 to obtain a sequence $H_{n,i}$ in V such that for all large n and all $i \leq n$, $h(n, i) < H_{n,i}(x_n)$. Let $\alpha_i = \sup_{n \geq i} [H_{n,i}]_{U_n} < j(\kappa^{+i})$. Find $\zeta < \mu^+$ such that for all $i < \omega$, $\alpha_\zeta^i > \alpha_i$. It is clear that for all large n and all $i \leq n$, $h(n, i) < H_{n,i}(x_n) < f_\zeta(n, i)$ using the fact that $[H_{n,i}]_{U_n} < \alpha_\zeta^i = j(g_{\alpha_\zeta}^i)(\sup(j^{\aleph_n} \cap j(\kappa^{+i})))$. \square

So $\langle f_\zeta \mid \zeta < \mu^+ \rangle$ is a scale.

Claim 4.5. *$\langle f_\zeta \mid \zeta < \mu^+ \rangle$ is very good.*

Again the proof is straightforward. Let $\gamma < \mu^+$ be such that $\omega < \text{cf}(\gamma) < \kappa$ in $V[G]$. So we can find a club $C \subseteq \gamma$ in V with $|C| < \kappa$. For all $\zeta < \zeta'$ from C we intersect (pointwise) the sequences of measure one sets which witness that $f_\zeta <^* f_{\zeta'}$. This is possible using κ -completeness of each U_n . The very goodness of γ follows from the fact that for all large n , x_n must come from this intersection.

We conclude the proof by noting that standard arguments show that $\langle f_\zeta \mid \zeta < \mu^+ \rangle$ can be collapsed to a scale in $\prod_{\substack{i \leq n \\ n < \omega}} \text{cf}(\sup(x_{n+1} \cap \kappa^{+i})) = \prod_{\substack{i \leq n \\ n < \omega}} \lambda_{n+1}^i$ with the same (modulo a club) very good points.

5. NO VERY GOOD SCALES IN $\prod_n \lambda_n^{+n+1}$ AND $\prod_n \lambda_n^{+n}$

In this section we prove the following theorem:

Theorem 5.1. *There is a condition p in \mathbb{P} which forces that there is no very good scale in $\prod_n \lambda_n^{+n+1}$.*

A similar theorem holds for the product $\prod_n \lambda_n^{+n}$. We also note that this theorem is also true in the context of the original preparation forcing. For the proof we modify an argument from [3] which shows that the existence of a very good scale is incompatible with simultaneous stationary reflection. We start with an analog of simultaneous reflection.

Proposition 5.2. *Suppose that $\langle S_n \mid n < \omega \rangle$ is a sequence of stationary sets in $\mu \cap \text{cof}(< \kappa)$ in $V[H^*]$. Then there is an inaccessible $\delta < \kappa$, such that the set*

$$\{\alpha \in \mu \cap \text{cof}(\delta^{++}) \mid (\forall n)(V[H^*] \models S_n \cap \alpha \text{ is stationary})\}$$

is stationary in $V[H^]$.*

Proof. κ remains supercompact in $V_0[H^*]$ since κ was indestructibly supercompact in V_0 . We may assume that E is generic over $V_0[H^*]$ for an Easton support iteration of α -directed closed forcing for $\alpha \leq \kappa$ inaccessible. It follows that we can force over $V_0[H^*][E]$ to lift an elementary embedding $j : V_0[H^*] \rightarrow N$ witnessing μ^+ -supercompactness of κ to $j : V_0[H^*][E] \rightarrow N^*$. Moreover the forcing to add the elementary embedding does not add any μ -sequences and hence preserves the stationarity of subsets of μ . Note that $V[H^*] = V_0[E][H^*] = V_0[H^*][E]$.

Let S_n for $n < \omega$ be a sequence of stationary subsets of $\mu \cap \text{cof}(< \kappa)$ in $V[H^*]$. Note that μ is collapsed in this model. Nevertheless if $\rho = \sup j^{\text{``}}\mu$, then $N^* \models j(S_n) \cap \rho$ is stationary for all n . The result follows by elementarity. \square

Proposition 5.3. *Suppose that X is a set of ordinals in $V[G]$ such that $\text{o.t.}(X) = \aleph_1$, then there is $Y \subseteq X$ in V which is unbounded in X .*

Proof. Let $X_p = \{\alpha \mid p \Vdash \alpha \in \dot{X}\}$. Let $q \in G$ be a condition of length at least 1. Then q fixes the parameter δ and forces $\delta^{++} = \aleph_1$. There are $n < \omega$ and $d \in \text{Coll}(\omega, \delta^+)_{V_0}$ such that

$$X' = \bigcup_{\{p \mid \text{lh}(p)=n, d_0^p=d\}} X_p$$

is unbounded in X . There is a name \dot{h} such that $q \Vdash \dot{h} : \delta^{++} \rightarrow X'$ is an increasing enumeration of X' . By extending q if necessary we may assume that for all $\alpha < \delta^{++}$, q satisfies the conclusion of the first part of the Prikry lemma, i.e. Lemma 2.6, applied with the dense set of conditions which decide the value of $\dot{h}(\alpha)$. We choose a condition p of length n extending q . Now p decides the values of \dot{h} up to an extension of the collapse conditions in the stem. Moreover we can leave the condition d fixed.

For all $\alpha < \delta^{++}$, we have that the set of $\langle d_1, c_0, \dots, c_{n-1} \rangle$ in

$$\begin{aligned} & \text{Coll}(\delta^{+3}, < \kappa_{x_0^p})_{V_0} \times \text{Coll}(\kappa_{x_0^p}^{++}, \kappa_{x_0^p}^{+\omega+1})_{V_0} \times \text{Coll}(\kappa_{x_0^p}^{+\omega+2}, < \kappa_{x_1^p})_{V_0} \times \dots \\ & \times \text{Coll}(\kappa_{x_{n-1}^p}^{+n+2}, \kappa_{x_{n-1}^p}^{+\omega+1})_{V_0} \times \text{Coll}(\kappa_{x_{n-1}^p}^{+\omega+2}, < \kappa)_{V_0} \end{aligned}$$

such that

$$\langle d, d_1, x_0^p, c_0, \dots, x_{n-1}^p, c_{n-1} \rangle \hat{\wedge} q \upharpoonright [n, \omega)$$

decides the value of $\dot{h}(\alpha)$ is dense open below $\langle d_1^p, c_0^p \dots c_{n-1}^p \rangle$. The distributivity of this poset is greater than δ^{++} . It follows that we can find an extension of p , which is in G and completely decides the values of \dot{h} . \square

In V , fix functions $\langle F_n^\beta \mid \beta < \kappa^{+\omega}, n < \omega \rangle$, such that each $F_n^\beta : \mathcal{P}_\kappa(\kappa^{+n}) \rightarrow \kappa$ and $[F_n^\beta]_{U_n} = \beta$.

We are now ready to prove Theorem 5.1.

Proof. Recall that $\mu = \kappa^{+\omega+1}$, and suppose for contradiction that $\langle f_\alpha \mid \alpha < \mu \rangle$ is a very good scale in $\prod_n \lambda_n^{+n+1}$. Suppose further that this is forced by the empty condition. For all $\alpha < \mu$ let $\langle H_n^\alpha \mid n < \omega \rangle$ be as in the conclusion of Lemma 3.2 applied to \dot{f}_α . Also, let $\langle p_\alpha \mid \alpha < \mu \rangle$ in \mathbb{P}/H and $\langle b_\alpha \mid \alpha < \mu \rangle$ in H be the conditions as in the proof of Lemma 3.2, that are used to define the H_n^α 's. We may arrange that for each $\alpha, n < \omega, b_\alpha(n) \leq_{C_n} [C_n^{p_\alpha}]_{U_n}$.

For each n , let H_n be \mathbb{D}_n -generic obtained from H^* . In particular, we have $V[H] \subset V[H_n] \subset V[H^*]$. Work in $V[H^*]$. For each α , let $n_\alpha < \omega$ be such that for all $n \geq n_\alpha, \langle b_\alpha(n) \mid n \geq n_\alpha \rangle \in H_{n_\alpha}$. By Fodor, there is some \bar{n} , such that for stationary many (in $V[H^*]$) $\alpha < \mu$ with countable cofinality, $\bar{n} = n_\alpha$. I.e. $S := \{\alpha \in \mu \cap \text{cof}(\omega) \mid \langle b_\alpha(n) \mid n \geq \bar{n} \rangle \in H_{\bar{n}}\}$ is stationary. For all $n \geq \bar{n}$, in $V[H_n]$ define

$$B_n := \{\alpha \in \mu \cap \text{cof}(\omega) \mid \langle b_\alpha(k) \mid k \geq n \rangle \in H_n\}.$$

Each B_n is also stationary, $S \subset B_n$, and $B_n \subset B_{n+1}$.

For each $n \geq \bar{n}$, define $\phi_n : \mu \rightarrow \kappa^{+n+1}$ by

$$\phi_n(\alpha) = [H_n^\alpha]_{U_n}.$$

For each $n < \omega$, apply Fodor to ϕ_n in $V[H_n]$, to find a stationary set $S_n \subset B_n$ and $\beta_n < \kappa^{+n+1}$, such that $\phi_n \upharpoonright S_n = \{\beta_n\}$. Note that in $V[H_n]$, μ is collapsed to κ^{+n+2} , but cardinals and cofinalities up to κ^{+n+2} are preserved, so Fodor applies. Since points in S_n have countable cofinality and \mathbb{D}^*/H_n is countably closed in $V[H_n]$, we have that S_n is also stationary in $V[H^*]$.

Since $\langle \dot{f}_\alpha \mid \alpha < \mu \rangle$ is forced to be a scale, and \mathbb{P}/H has the μ -c.c., for some $\eta < \mu, 1 \Vdash_{\mathbb{P}/H} n \mapsto F_n^{\beta_n}(\dot{x}_n) <^* \dot{f}_\eta$.

Fix δ as in the conclusion of Proposition 5.2, and denote $X := \{\alpha \in \mu \cap \text{cof}(\delta^{++}) \setminus \eta \mid (\forall n)(V[H^*] \models S_n \cap \alpha \text{ is stationary})\} \in V[H^*]$. X is stationary in $V[H^*]$. Since \mathbb{P}/H has the μ -c.c., the set of very good points is club in $V[H]$, and so it is club in $V[H^*]$. Let $\alpha \in X$ be a very good point.

Fix a club $A \subset \alpha$ with $o.t.(A) = \text{cf}(\alpha)$ and $A \subset \alpha \setminus \eta$.

Claim 5.4. *There is $p \in \mathbb{P}/H$ with length 1 forcing that $\delta^{++} = \aleph_1$, such that for all $\bar{n} \leq n < \omega$, for all $\beta \in B_n \cap A$, for all $x \in A_n^p, C_n^p(x) \leq C_n^{p_\beta}(x), b_\beta(n)(x)$.*

Proof. For $n \geq \bar{n}$, let K_n be the generic for \mathbb{C}_n induced by H^* and let $d_n \in K_n$ be a lower bound of $\{b_\beta(n) \mid \beta \in B_n \cap A\}$. Denote $d_n = [x \mapsto d_n(x)]_{U_n}$. Now by shrinking measure one sets, find $A_n \in U_n$, such that:

- $A_n \subset \bigcap_{\beta \in A} A_n^{p_\beta}$,
- for all $\beta \in A \cap B_n$, for all $x \in A_n$, $d_n(x) \leq C_n^{p_\beta}(x), b_\beta(n)(x)$.

Then define p by setting $A_n^p = A_n$ and $C_n^p(x) = d_n(x)$. Since $\langle d_n \mid \bar{n} \leq n < \omega \rangle \in H_{\bar{n}}$, we have that $p \in \mathbb{P}/H$. Also define d^p , so that $p \Vdash (\delta^{++})^V = \aleph_1$. \square

Let p be as in the above claim. By shrinking each A_n^p , we may assume that for all n , $A_n^p \subset \{x \in \mathcal{P}_\kappa(\kappa^{+n}) \mid (\forall \beta \in A) F_n^{[H_n^\beta]}(x) = H_n^\beta(x)\}$. Then for all $n \geq \bar{n}$, if $\beta \in A \cap S_n$ and $x \in A_n^p$, we have $F_n^{\beta n}(x) = H_n^\beta(x)$. Let G be \mathbb{P}/H -generic, such that $p \in G$.

Let $B \subset \alpha$ witness very goodness, $B \in V[G]$ with $o.t.(B) = \text{cf}(\alpha) = \aleph_1$. Then by Proposition 5.3, there is an unbounded $B' \subset B$ in V . By shrinking A if necessary, assume $A \subset \lim(B')$. In particular, A witnesses that α is very good.

Let $\bar{\beta} \in A$ and n be such that $F_n^{\bar{\beta} n}(x_n) < f_{\bar{\beta}}(n)$ and $\langle f_\beta(n) \mid \beta \in A \rangle$ are strictly increasing. Let $\beta \in A \cap S_n$, $\bar{\beta} < \beta$. Since $S_n \subset B_n$, by Remark 3.6 we have that $p \Vdash_{\mathbb{P}/H} f_{\bar{\beta}}(n) < H_n^\beta(x_n)$. So, in $V[G]$, we have

$$F_n^{\beta n}(x_n) < f_{\bar{\beta}}(n) < f_\beta(n) < H_n^\beta(x_n) = F_n^{\beta n}(x_n).$$

Contradiction. \square

Corollary 5.5. *A condition p as in Theorem 5.1 also forces $\square_{\kappa, \lambda}$ fails in $V[G]$ for all $\lambda < \kappa$.*

This is immediate from a theorem of Cummings, Foreman and Magidor [3], which shows that $\square_{\kappa, \lambda}$ implies the existence of very good scales.

6. A VERY GOOD SCALE IN $\prod_n \lambda_n^{+n+2}$

Recall that $\langle F_\beta \mid \beta < \mu \rangle$ in V are such that each $F_\beta : \kappa \rightarrow \kappa$ and $j_U(F_\beta)(\kappa) = \beta$. Here U is the top measure on $\mathcal{P}_\kappa(\mu)$. In $V[G]$, let $C_n \subset (\lambda_n^{+\omega+1})^V$ be a club of order type λ_n^{+n+2} added by the generic object for \mathbb{C}_{x_n} . Define $X' := \{\gamma < \mu \mid (\exists k)(\forall n \geq k) F_\gamma(\lambda_n) \in \lim(C_n)\}$. For each n , let \dot{C}_n^* be the canonical \mathbb{C}_n -name for the club of order type κ^{+n+2} in $\kappa^{+\omega+1}$. Define $X := \{\gamma < \mu \mid \exists [\vec{d}] \in H \text{ (for all large } n, d(n) \Vdash \gamma \in \lim(\dot{C}_n^*))\}$.

Lemma 6.1. *X' is equal to the set X .*

Proof. Suppose $\gamma \in X$. Then there is a sequence \vec{d} such that $[\vec{d}] \in H$ and for all large n , $d(n)$ forces γ is a limit point of \dot{C}_n^* . It follows that for all large n there is a measure one set of x , such that $d(n)(x)$ forces $F_\gamma(\kappa_x) \in \lim(\dot{C}_x)$, where \dot{C}_x is the canonical \mathbb{C}_x -name for a club of order type κ_x^{+n+2} in $\kappa_x^{+\omega+1}$. By the Prikry genericity of $\langle x_n \mid n < \omega \rangle$, it follows

that for all large n , $d(n)(x_n)$ forces $F_\gamma(\lambda_n) \in \lim(\dot{C}_{x_n})$. But since $[\vec{d}] \in H$, there is some $p \in G$, such that for all large n , $d(n)(x_n) = C_n^p(x_n)$. Then for all large n , $F_\gamma(\lambda_n) \in \lim(C_n)$, and so $\gamma \in X'$.

Let $\gamma \in X'$. Let $p \in G$ witness this. We may assume that there is $k < \omega$ such that for all $n > k$ and for all $x \in A_n^p$, $C_n^p(x)$ forces that $F_\gamma(\kappa_x)$ is a limit point of \dot{C}_x . It follows that $[\vec{C}^p] \in H$ and witnesses that $\gamma \in X$. \square

In [19], it was shown that X is $> \omega$ club. Define $\langle f_\gamma \mid \gamma \in X \rangle$ in $\prod_n \lambda_n^{+n+2}$, by

$$f_\gamma(n) = o.t.(C_n \cap F_\gamma(\lambda_n)).$$

We will show that this is a very good scale. If $\alpha < \beta$ are both in X , then for all large n , $F_\alpha(\lambda_n) < F_\beta(\lambda_n)$, and so for all large n , $f_\alpha(n) = o.t.(C_n \cap F_\alpha(\lambda_n)) < o.t.(C_n \cap F_\beta(\lambda_n)) = f_\beta(n)$. So, the sequence is increasing. Next we show that $\langle f_\gamma \mid \gamma \in X \rangle$ is cofinal. In $V[H^*]$, for every n , let $C_n^* \subset \mu$ be club of order type $(\kappa^{+n+2})^V$ added by the generic for \mathbb{C}_n . Using genericity of G , we arrange that if $\dot{C}_n^* = [x \rightarrow \dot{C}_x]_{U_n}$ is the canonical name, then for all large n , $\dot{C}_n = \dot{C}_{x_n}$.

Suppose that $h \in \prod_n \lambda_n^{+n+2}$. Let $\langle \gamma_n \mid n < \omega \rangle \in V[H^*]$ be given by Lemma 3.8 for h . I.e. every $\gamma_n < (\kappa^{+n+2})^V$ and for all large n , $h(n) < F_{\gamma_n}(\lambda_n)$. For every n , let $\gamma_n^* < \mu$ be the γ_n -th point of C_n^* .

Claim 6.2. *For all large n , $h(n) < o.t.(C_n \cap F_{\gamma_n^*}(\lambda_n))$.*

Proof. Let $p \in \mathbb{P}/H$ be such that for all large n ,

$$[C_n^p]_{U_n} \Vdash_{\mathbb{C}_n} \gamma_n = o.t.(\dot{C}_n^* \cap \gamma_n^*).$$

Then for all large n , for almost all x , $C_n^p(x) \Vdash_{\mathbb{C}_x} F_{\gamma_n}(\kappa_x) = o.t.(\dot{C}_x \cap F_{\gamma_n^*}(\kappa_x))$. So, for all large n , $F_{\gamma_n}(\lambda_n) = o.t.(C_n \cap F_{\gamma_n^*}(\lambda_n))$, and the claim follows. \square

Let $\gamma > \sup_n \gamma_n^*$, $\gamma \in X$. By intersecting measure one sets, we have that for all k , $\{x \mid \sup_n F_{\gamma_n^*}(\kappa_x) < F_\gamma(\kappa_x)\} \in U_k$. Then for all large n , $F_{\gamma_n^*}(\lambda_n) < F_\gamma(\lambda_n)$, so for all large n , $h(n) < o.t.(C_n \cap F_{\gamma_n^*}(\lambda_n)) < o.t.(C_n \cap F_\gamma(\lambda_n)) = f_\gamma(n)$. So $\langle f_\gamma \mid \gamma \in X \rangle$ is a scale.

To show it is very good, let $\gamma \in \lim(X)$ with $\text{cf}^{V[G]}(\gamma) > \omega$; we claim that γ is very good. There is some $n_\gamma < \omega$ and a condition $[\vec{d}] \in H$, such that for all $n > n_\gamma$, $d(n)$ forces γ to be a limit point of \dot{C}_n^* . Without loss of generality we take $Z_{\gamma,n}$ to be the value $\dot{C}_n^* \cap \gamma$ decided by $d(n)$. (We follow the notation used in the last section of [19].) Let $A \subset \gamma$ be a club of order type $\text{cf}^V(\gamma) = \text{cf}^{V[G]}(\gamma) < \kappa$, such that $A \subset \bigcap_{n \geq n_\gamma} Z_{\gamma,n}$, $A \in V$. By intersecting measure one sets, we have that for all large n :

- for all $\beta \in A$, $F_\beta(\lambda_n) \in C_n$,
- $\langle F_\beta(\lambda_n) \mid \beta \in A \rangle$ is increasing.

It follows that for all large n , $\langle o.t.(C_n \cap F_\beta(\lambda_n)) \mid \beta \in A \rangle$ is increasing i.e. $\langle f_\beta(n) \mid \beta \in A \rangle$ is increasing. So, every point in $\lim(X)$ of uncountable cofinality is very good.

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