ITERATING ALONG A PRIKRY SEQUENCE

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Abstract. In this paper we introduce a new method which combines Prikry forcing with an iteration between the Prikry points. Using our method we prove from large cardinals that it is consistent that the tree property holds at $\aleph_n$ for $n \geq 2$, $\aleph_\omega$ is strong limit and $2^{\aleph_\omega} = \aleph_{\omega+2}$.

The typical method for interleaving collapses with a Prikry sequence is to use a *product* of collapses. The original paper using this method is Magidor's paper [8] where he obtains the failure of the Singular Cardinals Hypothesis (SCH) at $\aleph_\omega$. There are many generalizations and variations on this method, examples include [9, 5, 6, 3]. In this paper we introduce a method for replacing the usual product with an iteration.

Broadly speaking Prikry forcing is a poset of finite approximations to a witness that some large cardinal κ (and perhaps some cardinals above κ) is singular of cofinality ω. The finite approximations are often called the *stem* of a condition. To avoid collapsing κ, the growth of the stem is controlled by measure one sets from some appropriate measure. The key lemma which allows us to prove that κ is preserved is called the Prikry Lemma. The result of the forcing is a singular cardinal κ which is still large in the sense that it is a limit of inaccessible cardinals.

Collapses can be added to Prikry forcing in order to make κ in to a small cardinal like $\aleph_\omega$. In particular we add collapses between the ordinals in the stem. We cannot only use a finite support product of collapses, since this would collapse κ. The solution is to constrain the values of future collapse conditions. For this we define constraining functions whose domains are measure one sets and whose values are elements of collapsing posets. To extend a condition, we select an element x from a measure one set which we add to the stem and we select a collapse condition which is below the value of the constraining function at x. This allows us to recover a version of the Prikry Lemma and show that again κ is preserved and the collapses have their desired effect.

The usual method of adding collapses to a Prikry forcing uses a product of posets which collapse between elements of the stem. A trivial but important observation is that in the usual scenario the values of the constraining functions do not depend on the stem. In particular the collapses are defined in a uniform way and each condition only has one constraining function. When we seek to iterate between the elements of the stem, the value of a given constraining function must depend on the stem, because the stem determines the space from which the name for a constrained collapse condition is taken.

To deal with this our technique has a complex system of constraining functions rather than just a single constraining function. One could think of our forcing as

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resembling a tree Prikry forcing with interleaved collapses where associated with each stem we have a measure one set and a constraining function. However instead of doing this we have integrated functions and measure one sets into a single sequence which has a function of n variables for each n.

The motivation for this technique comes from an old question of Magidor, "Is it consistent that every regular cardinal greater than $\aleph_1$ has the tree property?" The question was formulated in light of Mitchell's result [11] and progress was made by Abraham [1] and Cummings and Foreman [4]. The current longest known intervals of regular cardinals which can consistently have the tree property are due to Neeman [12] and the author [14]. Neeman showed from countably many supercompact cardinals it is consistent to have the tree property at every regular cardinal on the interval $[\aleph_2, \aleph_{\omega+1}]$. In Neeman's model, $\aleph_\omega$ is strong limit. In [14] by forcing $\aleph_\omega$ not to be strong limit, we showed that the interval can be extended to $[\aleph_2, \aleph_{\omega\cdot 2})$. The argument uses Neeman's work in an essential way.

There is an important distinction between models where $\aleph_\omega$ is strong limit and models where it is not. We note that by an old theorem of Specker [13] if $2^{\aleph_\omega} = \aleph_{\omega+1}$, then there is a special $\aleph_{\omega+2}$-tree. So a model for a positive answer to Magidor's question where $\aleph_\omega$ is strong limit must also have the failure of the Singular Cardinals Hypothesis at $\aleph_\omega$. In search of a positive answer, we return to the method of Prikry forcing with interleaved collapses, but we wish to use collapses that will enforce the tree property at each $\aleph_n$.

The known methods for getting the tree property at each $\aleph_n$ for $n \geq 2$ are due to Cummings and Foreman [4] and Neeman [12]. Cummings and Foreman define a full support iteration with countably many stages and Neeman revises the Cummings and Foreman iteration to make it act somewhat like a product. Both methods either explicitly or implicitly require an iteration. A recent paper of Friedman and Honzik [7] obtains the tree property at each $\aleph_{2n}$ for $n \geq 1$ and the failure of the Singular Cardinals Hypothesis at $\aleph_\omega$. Their method uses a product of collapses between the Prikry points, which as discussed above must look quite different from an iteration between the Prikry points.

So it appears that a model which gives a positive answer to Magidor's question with $\aleph_\omega$ strong limit must combine Prikry forcing with an iteration of some kind. In this paper we develop such a method and prove the following theorem.

Theorem 0.1. *Suppose there is a cardinal κ with an elementary embedding $j : V \to M$ witnessing that κ is huge with target λ and ${}^{\lambda^{++}}M \subseteq M$. There is a generic extension in which for all $n \geq 2$, the tree property holds at $\aleph_n$, $\aleph_\omega$ is strong limit and $2^{\aleph_\omega} = \aleph_{\omega+2}$.*

We use Cummings and Foreman's iteration for obtaining the tree property at each $\aleph_n$. For practical purposes, very little knowledge of the specifics of this iteration is required. Where we need a specific property of the iteration we cite the relevant lemma from the original paper. For completeness we recall that the iterates are posets of the form $\mathbb{R}(\tau, \kappa, V, W, F)$, which are κ-cc and designed to force $2^\tau = \kappa = \tau^{++}$ while preserving τ^+. Moreover if κ is supercompact and F is a Laver function for κ, then the tree property holds at κ in an indestructible way in the extension.

Barbanel [2] proved that from our large cardinal hypothesis one can obtain $2^\kappa = \kappa^{++}$ preserving the hugeness of κ. Working in this model we extract a measure which will be used in the definition of our Prikry forcing. Let $j : V \to M$ witness that κ is huge with target λ and assume that $2^\kappa = \kappa^{++}$. Let $\mu < j(\kappa)$ be a regular cardinal. It follows that the collection $U_\mu = \{A \subseteq \mathcal{P}_\kappa(\mu) \mid j``\mu \in j(A)\}$ is a supercompactness measure on $\mathcal{P}_\kappa(\mu)$. Moreover by the closure of M, $U_\mu \in M$. So in M we have for all $\mu < j(\kappa)$, κ is μ-supercompact. We define U to be U_{κ^+} and we note that the projection of U to a normal measure on κ concentrates on a set of cardinals which are $< \kappa$-supercompact.

1. Definition of the forcing

In this section we define the main forcing. Our Prikry forcing without the collapses will be the usual supercompact Prikry forcing from [8] defined with respect to the measure U.

Let $Z = \{x \in \mathcal{P}_\kappa(\kappa^+) \mid x \cap \kappa \in \kappa$ is $< \kappa$-supercompact and o.t.$(x) = (x \cap \kappa)^+\}$. Combining standard arguments with the argument at the end of the previous section, we have $Z \in U$. For $x \in Z$ we define $\kappa_x = x \cap \kappa$. Fix a class Laver function as in [10], that is a function $F : \mathrm{On} \to V$ such that if θ is a supercompact cardinal, then $F \restriction \theta$ is a Laver function for θ. We begin by defining a class of initial segments of Cummings-Foreman iterations. Let $\mathbb{P}(\emptyset)$ be the trivial forcing. Let $\vec{x} = \langle x_i \mid i < n\rangle$ be a supercompact Prikry stem from Z of length n, that is, a sequence of elements of Z which are increasing in the sense that if $i < j$, then $x_i \subseteq x_j$ and $|x_i| < x_j \cap \kappa$. Let $\mathbb{P}(\vec{x})$ be the first $n-1$ stages of a Cummings-Foreman iteration using the supercompact cardinals $\kappa_{x_0}, \dots \kappa_{x_{n-1}}$ and the Laver function F (see definitions 3.1 and 4.1 of [4]). Next we define a $\mathbb{P}(\vec{x})$-name for a poset $\dot{\mathbb{Q}}(\vec{x})$. There a few cases.

Case 1. $n = 0$

Then $\vec{x}$ is the empty stem and $\mathbb{P}(\vec{x})$ is the trivial forcing. In this case let $\mathbb{Q}(\emptyset)$ be $\mathbb{R}(\aleph_0, \kappa, V, V, F)$.

Case 2. $n = 1$

Let $\dot{\mathbb{Q}}(\vec{x})$ be the canonical $\mathbb{P}(\vec{x})$-name for $\mathbb{R}(\aleph_1^V, \kappa, V, V[\mathbb{P}(\vec{x})], F_{\vec{x}})$ where $F_{\vec{x}}$ is a function whose value at some α is the interpretation of $F(\alpha)$ if $F(\alpha)$ is a $\mathbb{P}(\vec{x})$-name and is 0 otherwise.

Case 3. $n \geq 2$

Let $\dot{\mathbb{Q}}(\vec{x})$ be the canonical $\mathbb{P}(\vec{x})$-term for the poset $\mathbb{R}(\kappa_{x_{n-2}}, \kappa, V[\mathbb{P}(\vec{x} \restriction n-1)], V[\mathbb{P}(\vec{x})], F_{\vec{x}})$ where $F_{\vec{x}}$ is a function defined as in Case 2.

This finishes the definition of the iteration part of the forcing. We prove a lemma which shows that the definition of the Prikry forcing is possible.

Lemma 1.1. *For every supercompact Prikry stem $\vec{x}$, for every $\vec{p} \in \mathbb{P}(\vec{x})$ and for every $\dot{q}$ such that $\Vdash_{\mathbb{P}(\vec{x})} \dot{q} \in \dot{\mathbb{Q}}(\vec{x})$, there is an $\alpha < \kappa$ such that for all $y \in Z$ with $\kappa_y > \alpha$, $\vec{p}^\frown\langle \dot{q}\rangle \in \mathbb{P}(\vec{x}^\frown y)$.*

Proof. Fix $\vec{x}, \vec{p}, \dot{q}$ as in the lemma. Note that in $V[\mathbb{P}(\vec{x})]$ the interpretation of $\dot{q}$ is a triple with each coordinate occupied by partial functions and that the union of the domains has size less than κ. Let τ be a $\mathbb{P}(\vec{x})$-name for the union of the domains. If $\vec{x} = \langle x_0, \dots x_{n-1}\rangle$, then $\mathbb{P}(\vec{x})$ has the $\kappa_{x_{n-1}}$-cc. It follows that in the ground model we can find α inaccessible so that $\vec{p} \Vdash_{\mathbb{P}(\vec{x})} \alpha > \sup \tau$. The claim follows from the uniformity of the definition of the forcings. We need to work through the cases of the definition of $\dot{\mathbb{Q}}(\vec{x})$.

Case 1. $n = 0$

In V for all $y \in Z$, we have $\mathbb{R}(\aleph_0, \kappa_y, V, V, F) \subseteq \mathbb{Q}(\emptyset) = \mathbb{R}(\aleph_0, \kappa, V, V, F)$. Now when $\kappa_y > \alpha$ we have that $\vec{p}^\frown\langle \dot{q}\rangle \in \mathbb{P}(\langle y\rangle)$.

Case 2. $n = 1$

In this case $\vec{x} = \langle x\rangle$. In $V[\mathbb{P}(\vec{x})]$ for all $y \in Z$ with $\kappa_y > \kappa_x$, we have $\mathbb{R}(\aleph_1^V, \kappa_y, V, V[\mathbb{P}(\vec{x})], F_{\vec{x}}) \subseteq \mathbb{Q}(\vec{x}) = \mathbb{R}(\aleph_1^V, \kappa, V, V[\mathbb{P}(\vec{x})], F_{\vec{x}})$. So if $\alpha < \kappa_y$, then $p^\frown\langle \dot{q}\rangle \in \mathbb{P}(\langle x, y\rangle)$.

Case 3. $n \geq 2$

In $V[\mathbb{P}(\vec{x})]$ for all $y \in Z$ with $\kappa_y > \kappa_{x_{n-1}}$, we have $\mathbb{R}(\kappa_{x_{n-2}}, \kappa_y, V[\mathbb{P}(\vec{x}) \restriction n-1], V[\mathbb{P}(\vec{x})], F_{\vec{x}}) \subseteq \mathbb{Q}(\vec{x}) = \mathbb{R}(\kappa_{x_{n-2}}, \kappa, V[\mathbb{P}(\vec{x}) \restriction n-1], V[\mathbb{P}(\vec{x})], F_{\vec{x}})$.

Note that in each case the forcing which we claim is a subset of $\mathbb{Q}(\vec{x})$ is exactly the top forcing in $\mathbb{P}(\vec{x}^\frown\langle y\rangle)$. Combining this with the choice of α finishes the proof. □

We are now ready to define the conditions of the main forcing. A condition is of the form

$$\langle x_0, p_0, x_1, p_1, \dots x_{n-1}, p_{n-1}, F_n, F_{n+1}, \dots\rangle$$

where $\vec{x} = \langle x_0 \dots x_{n-1}\rangle$ is a stem from the supercompact Prikry forcing, $\langle p_0, \dots p_{n-2}\rangle \in \mathbb{P}(\vec{x})$ and $\Vdash_{\mathbb{P}(\vec{x})} p_{n-1} \in \mathbb{Q}(\vec{x})$. There is an $X \in U$ whose powers form the domains of the functions F_{n+i} for $i < \omega$. In particular F_{n+i} is a function of $i + 1$ variables and has domain $[X]^{i+1}$ and for each $\vec{y} \in [X]^{i+1}$, $F_{n+i}(\vec{y})$ is a $\mathbb{P}(\vec{x}^\frown\vec{y})$ name for a condition in $\dot{\mathbb{Q}}(\vec{x}^\frown\vec{y})$. This finishes the definition of the conditions.

We will refer to conditions as $\langle \vec{s}, \vec{F}\rangle$ where

$$\vec{s} =_{def} \langle x_0, p_0, \dots x_{n-1}, p_{n-1}\rangle$$
$$\vec{F} =_{def} \langle F_n, F_{n+1}, \dots\rangle.$$

We define operators cp, pp so that $\mathrm{cp}(\vec{s}) = \langle p_0, \dots p_{n-1}\rangle$ and $\mathrm{pp}(\vec{s}) = \langle x_0, \dots x_{n-1}$. cp, pp stand for collapse part and Prikry part respectively. We write $\mathrm{dom}\,\vec{F}$ for the measure one set whose powers form the domains of the functions. We also call n the length of the condition $\langle \vec{s}, \vec{F}\rangle$ and sometimes denote it $\ell(\vec{s})$ or $\ell(\vec{x})$.

We move to the definition of the ordering.

$$\langle x_0, p_0, \dots x_{n-1}, p_{n-1}, F_n, F_{n+1}, \dots\rangle \geq$$
$$\langle x_0, q_0, \dots, x_{n-1}, q_{n-1}, x_n, q_n, \dots x_{m-1}, q_{m-1}, G_m, \dots\rangle$$

if and only if all of the following hold:

(1) For all $n \leq i < m$, $x_i \in \mathrm{dom}\,\vec{F}$ and $\mathrm{dom}\,\vec{G} \subseteq \mathrm{dom}\,\vec{F}$.
(2) $\langle p_0, p_1, \dots p_{n-1}, F_n(x_n), \dots F_{m-1}(\langle x_n, \dots x_{m-1}\rangle)\rangle \geq \langle q_0, \dots q_{m-1}\rangle$ in $\mathbb{P}(\langle x_0, \dots x_{m-1}\rangle) * \mathbb{Q}(\langle x_0, \dots x_{m-1}\rangle)$.
(3) For all $i < \omega$, and for all Prikry stems $\vec{y} \in [\mathrm{dom}\,\vec{G}]^i$,

$$\vec{q}^\frown\langle 1\rangle \Vdash_{\mathbb{P}(\vec{x}^\frown\vec{y})} F_{m+i-1}(\langle x_n, \dots x_{m-1}\rangle^\frown\vec{y}) \geq G_{m+i-1}(\vec{y})$$

where $\langle 1\rangle$ is the sequence of top elements of a final segment of the coordinates in $\mathbb{P}(\vec{x}^\frown\vec{y})$.

We define a *direct extension* to be an extension that preserves the length of a condition. We will often need to refer the minimal possible constraints available for a given stem. Suppose $\vec{t}$ is the stem of some extension of $\langle \vec{s}, \vec{F} \rangle$, then we define $\vec{F} \restriction \vec{t}$ as follows. Let $\vec{x} = \mathrm{pp}(\vec{t}) \setminus \mathrm{pp}(\vec{s})$ and suppose that $\ell(\vec{x}) = n$. For Prikry stems $\vec{y}$ of length i from $\mathrm{dom}(\vec{F})$, we set $(\vec{F} \restriction \vec{t})_i(\vec{y}) = F_{n+i}(\vec{x}^\frown \vec{y})$. It is not hard to check that $\langle \vec{t}, \vec{F} \restriction \vec{t} \rangle \leq \langle \vec{s}, \vec{F} \rangle$.

It is straightforward to check that the ordering as defined is transitive. Suppose that we have

$$\begin{aligned}
&\langle x_0, p_0, \dots x_{n-1}, p_{n-1}, F_n, F_{n+1} \dots \rangle \geq \\
&\langle x_0, q_0, \dots x_{n-1}, q_{n-1}, x_n, q_n, \dots x_{m-1}, q_{m-1}, G_m, G_{m+1}, \dots \rangle \geq \\
&\langle x_0, r_0, \dots x_{n-1}, r_{n-1}, x_n, r_n, \dots x_{m-1}, r_{m-1}, x_m, r_m, \dots x_{l-1}, r_{l-1}, H_l, H_{l+1} \dots \rangle
\end{aligned}$$

For convenience we let $\vec{x} = \langle x_0, x_1, \dots x_{l-1} \rangle$. (1) is obvious, since this is exactly the condition that we have in the usual supercompact Prikry forcing. For (2), we want to see that

$$\langle p_0, p_1, \dots p_{n-1}, F_n(x_n), \dots F_{l-1}(\vec{x} \restriction [n, l)) \rangle \geq \langle r_0, r_1, \dots r_{l-1} \rangle$$

in the poset $\mathbb{P}(\vec{x}) * \mathbb{Q}(\vec{x})$. To do this we insert a collapse condition given by the second condition. It is enough to show

$$\begin{aligned}
&\langle p_0, p_1, \dots p_{n-1}, F_n(x_n), \dots F_{l-1}(\vec{x} \restriction [n, l)) \rangle \geq \\
&\langle q_0, q_1, \dots, q_{m-1}, G_m(x_m), \dots G_{l-1}(\vec{x} \restriction [m, l)) \rangle \geq \\
&\langle r_0, r_1, \dots r_{l-1} \rangle
\end{aligned}$$

For the first $\geq$ we use both (2) and (3) from the definition. By (2) the first $\geq$ holds between the conditions restricted to m and by (3) and an easy induction it holds for the rest. The second $\geq$ is exactly (2) from the definition of the ordering. So we have shown (2) for the desired conditions.

For (3) we have that for all $\vec{y} \in [\mathrm{dom}(\vec{G})]^{i+1}$,

$$\vec{q}^\frown \langle 1 \rangle \Vdash F_{m+i}(\langle x_n, \dots x_{m-1} \rangle^\frown \vec{y}) \geq G_{m+i}(\vec{y})$$

and for all $\vec{z} \in [\mathrm{dom}(\vec{H})]^{k+1}$,

$$\vec{r}^\frown \langle 1 \rangle \Vdash G_{l+k}(\langle x_m, \dots x_{l-1} \rangle^\frown \vec{z}) \geq H_{l+k}(\vec{z})$$

We fix $\vec{z} \in [\mathrm{dom}(\vec{H})]^{k+1}$. We set $\vec{y} =_{def} \langle x_m, \dots x_{l-1} \rangle^\frown \vec{z}$. We apply the above inequalities for our fixed $\vec{z}, \vec{y}$. Note that $\langle x_n, \dots x_{m-1} \rangle^\frown \vec{y} = \langle x_n, \dots x_{l-1} \rangle^\frown \vec{z}$ and that $\vec{r}^\frown \langle 1 \rangle \leq \vec{q}^\frown \langle 1 \rangle$. (In each we adjoin a different number of 1's.) It follows from the transitivity of $\dot{\mathbb{Q}}(\vec{x}^\frown \vec{z})$ in $V[\mathbb{P}(\vec{x}^\frown \vec{z})]$ that

$$\vec{r}^\frown \langle 1 \rangle \Vdash F_{l+k}(\langle x_n, \dots x_{l-1} \rangle^\frown \vec{z}) \geq H_{l+k}(\vec{z})$$

which is what we wanted.

2. The Prikry Lemma

In this section we prove a version of the Prikry Lemma for our poset. Typical proofs of the Prikry lemma for forcing with interleaved collapses use closure (or at least distributivity) of the collapses to diagonalize over possible stems. This method is not possible for us, since our collapses are not closed enough. We get around this by carefully constructing names in our iteration between the Prikry points.

Lemma 2.1. *Fix a condition in the forcing $\langle \vec{s}, \vec{F} \rangle$ and a statement in the forcing language φ. There is a direct extension of $\langle \vec{s}, \vec{F} \rangle$ that decides φ.*

Proof. Fix $\langle \vec{s}, \vec{F} \rangle$ and φ as in the lemma. We let $\mathrm{pp}(\vec{s}) = \vec{x}$ and $\mathrm{cp}(\vec{s}) = \vec{p}$. The argument proceeds in three rounds. For each round we have a claim that provides a direct extension. Fix an enumeration of supercompact Prikry stems $\langle \vec{x}_\alpha \mid \alpha < \kappa^+ \rangle$ with the property that for all α if $\vec{x}_\alpha$ is a stem of length n then for all $m < n$ there is a $\beta < \alpha$ such that $\vec{x}_\alpha \restriction m = \vec{x}_\beta$.

Claim 2.2. *There is $\langle \vec{s}, \vec{G} \rangle \leq \langle \vec{s}, \vec{F} \rangle$ such that if $\langle \vec{t}, \vec{G}' \rangle \leq \langle \vec{s}, \vec{G} \rangle$ decides φ then there is a $\vec{u}$ such that $\ell(\vec{u}) = \ell(\vec{t})$, $\mathrm{pp}(\vec{u}) = \mathrm{pp}(\vec{t})$, $\mathrm{cp}(\vec{u}) \leq \mathrm{cp}(\vec{t})$ and $\langle \vec{u}, \vec{G} \restriction \vec{u} \rangle$ decides φ in the same way as $\langle \vec{t}, \vec{G}' \rangle$.*

We build a sequence of constraints $\langle \vec{F}^\alpha \mid \alpha < \kappa^+ \rangle$ which are suitable for $\vec{s}$ and so that $\langle \langle \vec{s}, \vec{F}^\alpha \rangle \mid \alpha < \kappa^+ \rangle$ is a decreasing sequence in the forcing. We also record measure one sets X_α in order to take a diagonal intersection at the end.

We go by induction on $\alpha < \kappa^+$. Define $\vec{F}^0 = \vec{F}$. Assume that we have constructed $\vec{F}^\alpha$ for some $\alpha < \kappa^+$. Consider the set

$$\begin{aligned} B_\alpha =_{def} \{ \vec{p} \in \mathbb{P}(\vec{x}_\alpha) * \dot{\mathbb{Q}}(\vec{x}_\alpha) \mid & \text{ there is a stem } \vec{t}_{\vec{p}} \text{ with } \mathrm{pp}(\vec{t}_{\vec{p}}) = \vec{x}_\alpha, \\ & \mathrm{cp}(\vec{t}_{\vec{p}}) = \vec{p} \text{ and there is a system of constraints } \vec{F}_{\vec{p}} \text{ such that} \\ & \langle \vec{t}_{\vec{p}}, \vec{F}_{\vec{p}} \rangle \leq \langle \vec{s}, \vec{F}^\alpha \rangle \text{ decides } \varphi \} \end{aligned}$$

Choose an antichain A'_α that is contained in B_α and is maximal. Extend A'_α to a maximal antichain A_α in $\mathbb{P}(\vec{x}_\alpha) * \dot{\mathbb{Q}}(\vec{x}_\alpha)$. By the κ-cc of this forcing, $|A_\alpha| < \kappa$. For each $\vec{p} \in A_\alpha$ fix a system of constraints $\vec{F}_{\vec{p}}$ witnessing that $\vec{p} \in B_\alpha$ if possible and otherwise let $\vec{F}_{\vec{p}} = \vec{F}_\alpha \restriction \vec{t}_{\vec{p}}$ where $\vec{t}_{\vec{p}}$ is determined by $\mathrm{pp}(\vec{t}_{\vec{p}}) = \vec{x}_\alpha$ and $\mathrm{cp}(\vec{t}_{\vec{p}}) = \vec{p}$.

Now that the reader has a feel for the forcing we make a small, but helpful notational change. From here on we will write things like $\vec{F}(\vec{y})$ to denote the constraint value for stem $\vec{y}$ when plugged into the appropriate function from $\vec{F}$. There is no risk of confusion and we alleviate the clutter of subscripts. We define the constraint functions $\vec{F}^{\alpha+1}$ as follows. We leave the domain unchanged, but record the set $X_\alpha =_{def} \bigcap_{\vec{p} \in A_\alpha} \mathrm{dom}(\vec{F}_{\vec{p}})$ which is in U by κ-completeness. For every $\vec{y}$ from X_α we define the term $\vec{F}^{\alpha+1}((\vec{x}_\alpha \setminus \vec{x})^\frown \vec{y})$ as follows (recall that $\vec{x}$ is $\mathrm{pp}(\vec{s})$). For each $\vec{p} \in A_\alpha$

$$\vec{p}^\frown \langle 1 \rangle \Vdash \vec{F}^{\alpha+1}((\vec{x}_\alpha \setminus \vec{x})^\frown \vec{y}) = \vec{F}_{\vec{p}}(\vec{y})$$

For all other $\vec{z}$ we let $\vec{F}^{\alpha+1}(\vec{z}) = \vec{F}^\alpha(\vec{z})$. In order to show that $\langle \vec{s}, \vec{F}^{\alpha+1} \rangle \leq \langle \vec{s}, \vec{F}^\alpha \rangle$, we need to show that for all relevant $\vec{z}$, $\mathrm{cp}(\vec{s}) \Vdash \vec{F}^{\alpha+1}(\vec{z}) \leq \vec{F}^\alpha(\vec{z})$. The nontrivial case is when $\vec{z} = (\vec{x}_\alpha \setminus \vec{x})^\frown \vec{y}$ for some $\vec{y}$ from X_α. Choose a $\mathbb{P}(\vec{x}_\alpha^\frown \vec{y})$-generic G so that $\mathrm{cp}(\vec{s}) \in G \restriction n$. Now A_α is a maximal antichain so we have $\vec{p} \in G \restriction \ell(\vec{x}_\alpha) \cap A_\alpha$ and hence in $V[G]$, $\vec{F}^{\alpha+1}(\vec{z}) = \vec{F}_{\vec{p}}(\vec{y})$. Now there are two cases. If $\vec{p} \notin B_\alpha$, then in $V[G]$ $\vec{F}_{\vec{p}}(\vec{y}) = \vec{F}^{\alpha}((\vec{x}_\alpha \setminus \vec{x})^\frown \vec{y})$, which is certainly enough. If $\vec{p} \in B_\alpha$, then in $V[G]$ we have $\vec{F}_{\vec{p}}(\vec{y}) \leq \vec{F}^{\alpha}((\vec{x}_\alpha \setminus \vec{x})^\frown \vec{y})$ by the choice of $\vec{F}_{\vec{p}}$. So in either case we are done. This completes the successor step of the construction.

Suppose that γ is a limit ordinal and we have constructed $\vec{F}^\alpha$ for all $\alpha < \gamma$. By construction all of the domains of functions remain the same. So for every $\vec{y}$ from the common domain we have that $\vec{s}$ forces that $\langle \vec{F}^\alpha(\vec{y}) \mid \alpha < \gamma \rangle$ is a decreasing sequence. We claim that we can find a lower bound, because the sequence only decreases finitely many times. At stage $\alpha + 1$ we only decreased the constraint values for stems extending $\vec{x}_\alpha \setminus \vec{x}$. So if we decreased the constraint value at stage $\alpha + 1$, then we must have had $\vec{x}_\alpha \setminus \vec{x}$ is an initial segment of $\vec{y}$. Hence for each $\vec{y}$ we can find a term $\vec{F}^\gamma(\vec{y})$ that $\mathrm{cp}(\vec{s})$ forces to be a lower bound for the sequence $\langle \vec{F}^\alpha(\vec{y}) \mid \alpha < \gamma \rangle$.

By a similar argument can find a lower bound for $\langle \langle \vec{s}, \vec{F}^\alpha \rangle \mid \alpha < \kappa^+ \rangle$ with stem $\vec{s}$. We restrict the resulting constraint to domains formed from the set

$$X =_{def} \{x \in \mathcal{P}_\kappa(\kappa^+) \mid \text{ for all } \alpha < \kappa^+ \text{ if } \vec{x}_\alpha^\frown x \text{ is a stem, then } x \in X_\alpha\}$$

We call this constraint $\vec{G}$ and note that $\langle \vec{s}, \vec{G} \rangle$ is a condition. We are ready to finish the proof of the claim. Suppose that $\langle \vec{t}, \vec{G}' \rangle \leq \langle \vec{s}, \vec{G} \rangle$ decides φ. Then we have $\mathrm{pp}(\vec{t}) = \vec{x}_\alpha$ for some $\alpha < \kappa^+$. We claim that $\mathrm{cp}(\vec{t}) \in B_\alpha$ as witnessed by $\vec{t}$ and $\vec{G}'$. In particular we need to show that $\langle \vec{t}, \vec{G}' \rangle \leq \langle \vec{s}, \vec{F}^\alpha \rangle$. This is clear from the transitivity of the ordering, since $\langle \vec{s}, \vec{G} \rangle$ is between them. It follows that there is $\vec{p} \in A'_\alpha$ such that $\vec{p}$ is compatible with $\mathrm{cp}(\vec{t})$. Let $\vec{r}$ be a common extension and let $\vec{u}$ be the stem obtained from combining $\vec{x}_\alpha$ and $\vec{r}$. We claim that $\langle \vec{u}, \vec{G} \restriction \vec{u} \rangle$ decides φ in the same way that $\langle \vec{t}, \vec{G}' \rangle$ does.

By construction we have that there is a stem $\vec{t}_{\vec{p}}$ and a system of constraints $\vec{F}_{\vec{p}}$ chosen at the inductive step with $\mathrm{pp}(\vec{t}_{\vec{p}}) = \vec{x}_\alpha$ and $\mathrm{cp}(\vec{t}_{\vec{p}}) = \vec{p}$, such that $\langle \vec{t}_{\vec{p}}, \vec{F}_{\vec{p}} \rangle \leq \langle \vec{s}, \vec{F}^\alpha \rangle$ decides φ. We claim that $\langle \vec{u}, \vec{G} \restriction \vec{u} \rangle \leq \langle \vec{t}_{\vec{p}}, \vec{F}_{\vec{p}} \rangle$. We check conditions $1-3$ in the definition of the ordering. The conditions have the same length, so for (1) it suffices to check that $\mathrm{dom}(\vec{G} \restriction \vec{u}) \subseteq \mathrm{dom}(\vec{F}_{\vec{p}})$. Suppose that $x \in \mathrm{dom}(\vec{G} \restriction \vec{u})$. We have that $x \in X$. Since in the definition of a condition we have that $\vec{x}_\alpha^\frown x$ is a stem, it follows that $x \in X_\alpha$. We are done since $\mathrm{dom}(\vec{F}_{\vec{p}}) \supseteq X_\alpha$. For condition (2), we have that $\mathrm{cp}(\vec{t}_{\vec{p}}) = \vec{p} \geq \vec{r} = \mathrm{cp}(\vec{u})$ and this is enough since the conditions have the same length. For condition (3) we need that for all $\vec{y}$ from the $\mathrm{dom}(\vec{G} \restriction \vec{u})$, $\mathrm{cp}(\vec{u})$ forces $(\vec{G} \restriction \vec{u})(\vec{y}) \leq \vec{F}_{\vec{p}}(\vec{y})$. This follows from collected facts about the construction of $\vec{G}$.

$$\begin{aligned}
&(\vec{G} \restriction \vec{u})(\vec{y}) = \vec{G}((\vec{x}_\alpha \setminus \vec{x})^\frown \vec{y}) \\
&\vec{p} \Vdash \vec{F}_{\vec{p}}(\vec{y}) = \vec{F}^{\alpha+1}((\vec{x}_\alpha \setminus \vec{x})^\frown \vec{y}) \\
&\mathrm{cp}(\vec{s}) \Vdash \vec{F}^{\alpha+1}((\vec{x}_\alpha \setminus \vec{x})^\frown \vec{y}) \geq \vec{G}((\vec{x}_\alpha \setminus \vec{x})^\frown \vec{y})
\end{aligned}$$

To finish the proof we note that $\vec{p} \leq \mathrm{cp}(\vec{s})^\frown \langle 1 \rangle$ and $\vec{r} \leq \vec{p}$. It follows that $\langle \vec{u}, \vec{G} \restriction \vec{u} \rangle$ decides φ. We would like to see that $\langle \vec{u}, \vec{G} \restriction \vec{u} \rangle$ and $\langle \vec{t}, \vec{G}' \rangle$ give the same decision. To do this we note that they are compatible, since $\langle \vec{u}, \vec{G}' \restriction \vec{u} \rangle$ is below both. Running through the definition of the ordering is routine. This finishes the first round of the construction.

Remark 2.3. *Every direct extension of $\langle \vec{s}, \vec{G} \rangle$ retains the same universal property.*

We turn to the second round of the construction, for which we have the following claim.

Claim 2.4. *There is a direct extension $\langle \vec{s}, \vec{H} \rangle \leq \langle \vec{s}, \vec{G} \rangle$ such that if an extension $\langle \vec{t}, \vec{H}' \rangle \leq \langle \vec{s}, \vec{H} \rangle$ decides φ, then there is a stem $\vec{v}$ such that*

(1) $\mathrm{pp}(\vec{t}) = \mathrm{pp}(\vec{v})$,
(2) $\mathrm{cp}(\vec{v}) \restriction (\ell(\vec{v}) - 1) \leq \mathrm{cp}(\vec{t}) \restriction (\ell(\vec{t}) - 1)$,
(3) $\mathrm{cp}(\vec{v})(\ell(\vec{v}) - 1) = \vec{H}(\mathrm{pp}(\vec{v}) \setminus \mathrm{pp}(\vec{s}))$ *and*
(4) $\langle \vec{v}, \vec{H} \restriction \vec{v} \rangle$ *decides φ in the same way as* $\langle \vec{t}, \vec{H}' \rangle$.

Recall that we enumerated the Prikry stems extending $\mathrm{pp}(\vec{s}) = \vec{x}$ as $\langle \vec{x}_\alpha \mid \alpha < \kappa^+ \rangle$. Our approach is similar to the previous claim. We construct a decreasing sequence of conditions $\langle \langle \vec{s}, \vec{G}^\alpha \rangle \mid \alpha < \kappa^+ \rangle$. To begin we fix $\vec{G}^0 = \vec{G}$.

Suppose that we have constructed $\vec{G}^\alpha$ for some $\alpha < \kappa^+$. Let

$$B_\alpha =_{def} \{\vec{p} \in \mathbb{P}(\vec{x}_\alpha) \mid \text{there are } \vec{u}, \dot{q} \text{ such that } \mathrm{cp}(\vec{u}) = \vec{p}^\frown \langle \dot{q} \rangle,$$
$$\mathrm{pp}(\vec{u}) = \vec{x}_\alpha,\ \vec{p} \Vdash \dot{q} \leq \vec{G}^\alpha(\vec{x}_\alpha \setminus \vec{x}) \text{ and } \langle \vec{u}, \vec{G}^\alpha \restriction \vec{u} \rangle \text{ decides } \varphi\}$$

Again we pick a maximal antichain $A'_\alpha \subseteq B_\alpha$ and then extend it to a maximal antichain $A_\alpha \subseteq \mathbb{P}(\vec{x}_\alpha)$. Note that this time we have $|A_\alpha| < \kappa_x$ where x is the top element of $\vec{x}_\alpha$. For each $\vec{p} \in A_\alpha$, if $\vec{p} \in B_\alpha$, then we let $\dot{q}_{\vec{p}}$ witness this. Otherwise we define $\dot{q}_{\vec{p}} = \vec{G}^\alpha(\vec{x}_\alpha \setminus \vec{x})$. Define $\vec{G}^{\alpha+1}$ as follows. Let $\vec{G}^{\alpha+1}(\vec{x}_\alpha \setminus \vec{x})$ to be a $\mathbb{P}(\vec{x}_\alpha)$-term such that for all $\vec{p} \in A_\alpha$, $\vec{p} \Vdash \vec{G}^{\alpha+1}(\vec{x}_\alpha \setminus \vec{x}) = \dot{q}_{\vec{p}}$. For all other $\vec{y}$ we let $\vec{G}^{\alpha+1}(\vec{y}) = \vec{G}^\alpha(\vec{y})$. We need to check that $\langle \vec{s}, \vec{G}^{\alpha+1} \rangle \leq \langle \vec{s}, \vec{G}^\alpha \rangle$. Conditions (1) and (2) are obvious. For (3) notice that the only interesting case is when $\vec{y} = \vec{x}_\alpha \setminus \vec{x}$. In this case we have a maximal antichain A_α such that for each $\vec{p} \in A_\alpha$, $\vec{p} \Vdash \vec{G}^{\alpha+1}(\vec{y}) \leq \vec{G}^\alpha(\vec{y})$. So in fact for all $\vec{y}$ it is forced that $\vec{G}^{\alpha+1}(\vec{y}) \leq \vec{G}^\alpha(\vec{y})$. Therefore $\vec{s}^\frown\langle 1 \rangle$ forces this.

Assume that γ is a limit ordinal. We can take a lower bound for the construction so far since for all $\vec{y}$ the sequence $\langle \vec{G}^\alpha(\vec{y}) \mid \alpha < \kappa^+ \rangle$ only decreases at stage α where $\vec{y} = \vec{x}_\alpha \setminus \vec{x}$. We can assume that all of our functions have the same domain and so we have a sequence of constraint functions defined on measure one sets, $\vec{G}^\gamma$.

A similar argument allows us to find a lower bound $\langle \vec{s}, \vec{H} \rangle$ for the whole construction. We will show that $\langle \vec{s}, \vec{H} \rangle$ is as required for the claim. Suppose that $\langle \vec{t}, \vec{H}' \rangle \leq \langle \vec{s}, \vec{H} \rangle$ decides φ. Then by the first claim we have that there is a $\vec{u}$ so that $\ell(\vec{u}) = \ell(\vec{t})$, $\mathrm{cp}(\vec{u}) \leq \mathrm{cp}(\vec{t})$ and $\langle \vec{u}, \vec{G} \restriction \vec{u} \rangle$ decides φ in the same way as $\langle \vec{t}, \vec{H}' \rangle$. Then there is an $\alpha < \kappa^+$ such that $\mathrm{pp}(\vec{u}) = \vec{x}_\alpha$. Let $\mathrm{cp}(\vec{u}) =_{def} \langle q_0, \dots q_{k-1} \rangle$. We claim that $\langle q_0, \dots q_{k-2} \rangle \in B_\alpha$ and that this is witnessed by $\vec{u}, q_{k-1}$ in place of $\vec{u}, \dot{q}$ respectively. Clearly we have that $\mathrm{cp}(\vec{u}) = \langle q_0, \dots q_{k-2} \rangle^\frown \langle q_{k-1} \rangle$. From the definition of the ordering we have

$$\langle q_0, \dots q_{k-2} \rangle \Vdash q_{k-1} \leq H(\vec{x}_\alpha \setminus \vec{x})$$
$$\mathrm{cp}(\vec{s})^\frown \langle 1 \rangle \Vdash \vec{H}(\vec{x}_\alpha \setminus \vec{x}) \leq \vec{G}^\alpha(\vec{x}_\alpha \setminus \vec{x})$$

It follows that $\langle q_0, \dots q_{k-2} \rangle \Vdash q_{k-1} \leq \vec{G}^\alpha(\vec{x}_\alpha \setminus \vec{x})$. To finish we recall that $\langle \vec{u}, \vec{G} \restriction \vec{u} \rangle$ decides φ. So by the choice of A'_α, there is a $\vec{p} \in A'_\alpha$, such that $\vec{p}$ is compatible with $\langle q_0, \dots q_{k-2} \rangle$. Let $\vec{u}_{\vec{p}}, \dot{q}_{\vec{p}}$ be the witnesses to the fact that $\vec{p} \in B_\alpha$. Choose $\vec{r}$

below both $\langle q_0, \dots q_{k-2}\rangle$ and $\vec{p}$. Let $\vec{v}$ be the stem determined by $\vec{r}^\frown\langle H(\vec{x}_\alpha \searrow \vec{x})\rangle$ and $\vec{x}_\alpha$. It is clear that parts $1-3$ of the claim are satisfied.

To complete the proof of the claim, we need to show that $\langle \vec{v}, \vec{H} \restriction \vec{v}\rangle$ decides φ in the same way as $\langle \vec{u}, \vec{G} \restriction \vec{u}\rangle$. We show that the above condition decides φ by showing that it is below $\langle \vec{u}_{\vec{p}}, \vec{G}^\alpha \restriction \vec{u}_{\vec{p}}\rangle$. Note that the first part of condition (1) is trivial since $\mathrm{pp}(\vec{u}_{\vec{p}}) = \mathrm{pp}(\vec{v})$. We leave the second part of condition (1) until after we have proved condition (2). Let $k = \ell(\vec{v})$. Then by the choice of $\vec{v}$ we have that $\mathrm{cp}(\vec{v}) \restriction k-1 = \vec{r} \leq \vec{p} = \mathrm{cp}(\vec{u}_{\vec{p}}) \restriction k-1$. To finish with condition (2) we need to show that $\vec{r} \Vdash \vec{H}(\vec{x}_\alpha \searrow \vec{x}) \leq \dot{q}_{\vec{p}}$. To do this we collect some facts about the construction of $\vec{G}^{\alpha+1}$.

$$\vec{p} \Vdash \vec{G}^{\alpha+1}(\vec{x}_\alpha \searrow \vec{x}) = \dot{q}_{\vec{p}}$$
$$\mathrm{cp}(\vec{s})^\frown\langle 1\rangle \Vdash \vec{H}(\vec{x}_\alpha \searrow \vec{x}) \leq \vec{G}^{\alpha+1}(\vec{x}_\alpha \searrow \vec{x})$$

This finishes condition (2), since $\vec{r}$ is below both $\vec{s}^\frown\langle 1\rangle$ and $\vec{p}$. It follows that $\mathrm{dom}(\vec{H} \restriction \vec{v}) \subseteq \mathrm{dom}(\vec{G}^\alpha) \restriction \vec{u}_{\vec{p}}$, since $\mathrm{dom}(\vec{H}) = \mathrm{dom}(\vec{G}^\alpha)$ and if $\vec{v}^\frown y$ is a stem, then $\vec{u}_{\vec{p}}^\frown y$ is a stem. Condition (3) follows from the fact that $\langle \vec{s}, \vec{H}\rangle \leq \langle \vec{s}, \vec{G}^\alpha\rangle$.

To see that $\langle \vec{v}, \vec{H} \restriction \vec{v}\rangle$ gives the same decision as $\langle \vec{u}, \vec{G} \restriction \vec{u}\rangle$ we show that they are compatible. To show this we need to see that $\vec{r} \Vdash q_{k-1} \leq \vec{H}(\vec{x}_\alpha \searrow \vec{x})$. Recall that in fact $\langle q_0, \dots q_{k-2}\rangle$ forces this and $\vec{r}$ is below it. So we can take the stem determined by $\vec{x}_\alpha$ and $\vec{r}^\frown\langle q_{k-1}\rangle$ together with $\vec{H}$ restricted to this stem and this is below both conditions. This finishes the second claim.

In the third round we restrict the measure one set to obtain the same decision. This will be enough to finish the proof of the Prikry Lemma. For the third and final round we actually need an enumeration of all stems $\langle \vec{s}_\alpha \mid \alpha < \kappa^+\rangle$ not just Prikry stems. For each $\alpha < \kappa^+$ we partition the set of $x \in Z$ such that $\vec{s}_\alpha^\frown x$ is a stem into three sets. For ease of notation we write $p(x)$ for $\vec{H}((\mathrm{pp}(\vec{s}_\alpha)^\frown x \setminus \vec{x})$.

$$Y^0_\alpha = \{x \in Z \mid \langle \vec{s}_\alpha^\frown\langle x, p(x)\rangle, \vec{H} \restriction (\vec{s}_\alpha^\frown\langle x, p(x)\rangle)\rangle \Vdash \varphi\}$$
$$Y^1_\alpha = \{x \in Z \mid \langle \vec{s}_\alpha^\frown\langle x, p(x)\rangle, \vec{H} \restriction (\vec{s}_\alpha^\frown\langle x, p(x)\rangle)\rangle \Vdash \neg\varphi\}$$
$$Y^2_\alpha = \{x \in Z \mid \langle \vec{s}_\alpha^\frown\langle x, p(x)\rangle, \vec{H} \restriction (\vec{s}_\alpha^\frown\langle x, p(x)\rangle)\rangle \nVdash \varphi\}$$

For each α there is $m_\alpha \in 3$ such that $Y^{m_\alpha}_\alpha \in U$. Let $Y = \{x \in Z \mid \forall\alpha < \kappa^+$ if $\vec{s}_\alpha^\frown x$ is a stem, then $x \in Y^{m_\alpha}_\alpha\}$. We have $Y \in U$ and we let $\vec{I}$ be the restriction of $\vec{H}$ to Y. Clearly $\langle \vec{s}, \vec{I}\rangle \leq \langle \vec{s}, \vec{H}\rangle$. We present a claim that finishes the proof.

Claim 2.5. *There is a direct extension of $\langle \vec{s}, \vec{I}\rangle$ that decides φ.*

We argue by contradiction. Suppose that no direct extension of $\langle \vec{s}, \vec{I}\rangle$ decides φ. Then any extension which decides φ must add at least one Prikry point. Let $\langle \vec{t}, \vec{I}'\rangle$ be an extension of minimal length that decides φ without loss of generality we assume that it forces φ. By the second round there is a stem $\vec{v}$ satisfying $1-4$ of the claim. There is an $\alpha < \kappa^+$ such that $\vec{v} \restriction (\ell(\vec{v}) - 1) = \vec{s}_\alpha$. From the definition of Y, we have that the top element of $\mathrm{pp}(\vec{v})$ is in $Y^{m_\alpha}_\alpha$. It follows from the conditions of the Claim 2.4 that $m_\alpha = 0$. Now we can define a condition of shorter length that forces φ, a contradiction. We claim that $\langle \vec{s}_\alpha, \vec{I} \restriction \vec{s}_\alpha\rangle$ forces φ. Every one step extension is below a condition of the form

$$\langle \vec{s}_\alpha^\frown x^\frown \vec{H}((\mathrm{pp}(\vec{s}_\alpha)^\frown x) \searrow \vec{x}), \vec{H} \restriction (\vec{s}_\alpha^\frown x^\frown \vec{H}((\mathrm{pp}(\vec{s}_\alpha)^\frown x) \searrow \vec{x}))\rangle$$

for some $x \in Y^0_\alpha$. Hence there is a dense set of conditions below $\langle \vec{s}_\alpha, \vec{I} \restriction \vec{s}_\alpha \rangle$ which force φ. This finishes the claim and with it the proof of the Prikry Lemma. $\square$

We need an additional argument to show that bounded subsets of κ come from initial segments of the generic. We recall a part of Lemma 4.3 from Cummings-Foreman that has been modified to fit our new context.

Lemma 2.6. *For all $n \geq 3$ if $\vec{x}$ is a Prikry stem of length n, then $V[\mathbb{P}(\vec{x})] \vDash \dot{\mathbb{Q}}(\vec{x})$ is $\aleph_{n-1}$-closed.*

For a stem of length $n \geq 2$, $\mathbb{P}(\vec{x})$ preserves $\aleph_0, \aleph_1$ and for $0 \leq i \leq n-1$ makes κ_{x_i} into $\aleph_{i+2}$. The following lemma is clear from the above facts.

Lemma 2.7. *Fix a Prikry stem $\vec{x}$ of length $n \geq 3$. Given a sequence of fewer than $\kappa_{\vec{x}(n-3)}$ many $\mathbb{P}(\vec{x})$-terms for conditions in $\dot{\mathbb{Q}}(\vec{x})$, if $\vec{p} \in \mathbb{P}(\vec{x})$ forces that they are a decreasing sequence, then there is a $\mathbb{P}(\vec{x})$-term that $\vec{p}$ forces to be a lower bound.*

From this we have the following lemma which is used in our lemma about bounded subsets.

Lemma 2.8. *Given a $\mu < \kappa$ and a condition $\langle \vec{s}, \vec{F} \rangle$. Then there is an extension $\langle \vec{s}, \vec{G} \rangle$ of $\langle \vec{s}, \vec{F} \rangle$ such that every extension of $\langle \vec{s}, \vec{G} \rangle$ which adds at least two Prikry points has the property that the term forcing in the constraints is μ^+-closed.*

Proof. To show this it suffices to shrink the measure one set of the condition $\langle \vec{s}, \vec{F} \rangle$. We restrict the domains of the $\vec{F}$ so that for all x in the restriction $\kappa_x > \mu^+$. This defines $\langle \vec{s}, \vec{G} \rangle$. Suppose that $\langle \vec{t}, \vec{H}' \rangle \leq \langle \vec{s}, \vec{H} \rangle$ is an extension that adds at least two Prikry points say they are x_n, x_{n+1}. By the choice of $\vec{G}$ we have $\mu^+ < \kappa_{x_n} < \kappa_{x_{n+1}}$. By the previous lemma we have that all of the term forcings in the constraints are at least κ_{x_n}-closed in the ground model. $\square$

Lemma 2.9. *Suppose that $\dot{b}$ is a name in the main forcing for a subset of some $\mu < \kappa$, then it is forced that there is an n such that $\dot{b}$ in $V[\mathbb{P}(\dot{\vec{x}} \restriction n)]$.*

Note that we used $\dot{\vec{x}}$ for the canonical name for the Prikry sequence.

Proof. By the previous lemma there is a dense set of conditions so that the forcing in the constraint is μ^+-closed. Suppose that $\langle \vec{s}, \vec{F} \rangle$ is a condition in this dense set. Let $\ell(\vec{s}) = k$, $\mathrm{pp}(\vec{s}) = \vec{x}$ and $\mathrm{cp}(\vec{s}) = \vec{p}$. We construct an extension $\langle \vec{s}, \vec{G} \rangle$, which forces that $\dot{b} \in V[\mathbb{P}(\dot{\vec{x}}) \restriction k+1]$. We construct a decreasing sequence of μ many conditions $\langle \langle \vec{s}, \vec{F}^\alpha \rangle \mid \alpha < \mu \rangle$. We let $\vec{F}^0 = \vec{F}$. At stage $\alpha + 1$ of the construction we repeatedly apply the Prikry lemma to the condition $\langle \vec{s}, \vec{F}^\alpha \rangle$ and the statement "$\alpha \in \dot{b}$" to obtain $\vec{F}^{\alpha+1}$ such that there are a maximal antichain of elements $\vec{q}$ of $\mathbb{P}(\vec{x}) * \mathbb{Q}(\vec{x})$ below $\vec{p}$ where if $\vec{t}$ is obtained from $\vec{x}$ and $\vec{q}$, then $\langle \vec{t}, \vec{F}^{\alpha+1} \rangle$ decides "$\alpha \in \dot{b}$". As in the proof of Claim 2.2 we amalgamate different extensions of $\vec{F}^\alpha$ over the maximal antichain in $\mathbb{P}(\vec{x}) * \mathbb{Q}(\vec{x})$ to obtain $\vec{F}^{\alpha+1}$.

The closure of the forcing in the constraining functions allows us to take a lower-bound at limit stages of the construction and also for the whole sequence of conditions. We call this lowerbound $\langle \vec{s}, \vec{G} \rangle$. Clearly this condition forces that $\dot{b}$ is in the extension $V[\mathbb{P}(\dot{\vec{x}} \restriction k+1)]$. $\square$

Corollary 2.10. *In the extension $\kappa = \aleph_\omega$ and for all $n \geq 2$, the tree property holds at $\aleph_n$.*

This follows easily from facts about the original Cummings-Foreman model and the previous lemma.

3. Cardinals above κ

In this section we seek to show that the technique from Magidor's paper [8] is enough to give us the failure of SCH at $\aleph_\omega$ with our forcing. In particular we show that an inner model of the full extension preserves cardinals above κ.

We need the following variation of Claim 2.2. Fix a condition $\langle \vec{s}, \vec{F} \rangle$ and suppose that $\dot{\gamma}$ is a name for an ordinal. We claim that there is a direct extension $\langle \vec{s}, \vec{G} \rangle$ such that if $\langle \vec{t}, \vec{G}' \rangle \leq \langle \vec{s}, \vec{G} \rangle$ forces the value of $\dot{\gamma}$ to be γ, then so does $\langle \vec{t}, \vec{G} \restriction \vec{t} \rangle$. To see this we repeat the proof of Claim 2.2, but instead of deciding φ, we decide the value of $\dot{\gamma}$ if possible. The proof goes through because $\langle \vec{t}, \vec{G}' \rangle$ and $\langle \vec{t}, \vec{G} \restriction \vec{t} \rangle$ are compatible and so must force the same value for $\dot{\gamma}$.

Let G be generic for our forcing. We are interested in the inner model of $V[G]$ corresponding to the measurable Prikry sequence and the generics generated by the Cummings-Foreman iteration. In particular if $\langle x_n \mid n < \omega \rangle$ is the Prikry sequence and g_ω is the sequence of generics such that $g_\omega \restriction n$ is generic for $\mathbb{P}(\vec{x} \restriction n+1)$ obtained from G, then we are interested in $V_0 =_{def} V[\langle \kappa_{x_n} \mid n < \omega \rangle, g_\omega]$. Note that Lemma 2.9 gives that in V_0, $\kappa = \aleph_\omega$ and for all $n \geq 2$, the tree property holds at $\aleph_n$.

We prove the following theorem.

Theorem 3.1. *The cardinals κ^+ and κ^{++} are preserved in V_0.*

We will do this by analyzing automorphisms of our forcing. Suppose that Γ is an permutation of κ^+ that fixes κ. Then we can apply Γ to a condition $\langle \vec{s}, \vec{F} \rangle =_{def} \langle x_0, p_0, \dots, x_{n-1}, p_{n-1}, F_n, F_{n+1} \dots \rangle$, as follows.

$$\Gamma(\langle \vec{s}, \vec{F} \rangle) = \langle \Gamma``x_0, p_0, \dots \Gamma``x_{n-1}, p_{n-1}, F_n \circ \Gamma^{-1}, \dots F_{n+m} \circ \oplus_{i<m} \Gamma^{-1} \dots \rangle$$

Note that $\Gamma(\langle \vec{s}, \vec{F} \rangle)$ is a condition since Γ fixes κ. It is easy to see that Γ is an automorphism of the forcing and that any name for $\langle \kappa_n \mid n < \omega \rangle$ and g_ω is fixed by Γ.

Straight from [8] we have the following lemma.

Lemma 3.2. *Let Γ be a permutation of α and U be a normal fine measure on $\mathcal{P}_\kappa(\alpha)$, then $\{x \mid \Gamma``x = x\} \in U$.*

Lemma 3.3. *Let*

$$\langle \vec{s}, \vec{F} \rangle = \langle x_0, p_0, \dots, x_{n-1}, p_{n-1}, F_n, F_{n+1} \dots \rangle$$
$$\langle \vec{t}, \vec{G} \rangle = \langle y_0, p_0, \dots, y_{n-1}, p_{n-1}, G_n, G_{n+1} \dots \rangle$$

where $\kappa_{x_i} = \kappa_{y_i}$ for all $i < n$ and the constraint functions agree on the intersection of their domains. Then there is a permutation Γ of κ^+ which fixes κ such that $\Gamma(\langle \vec{s}, \vec{F} \rangle)$ is compatible with $\langle \vec{t}, \vec{G} \rangle$.

Proof. Recall that for all $x \in Z$, $|x| = (x \cap \kappa)^+$. An easy inductive construction works to find Γ such that $\Gamma \restriction x_i : x_i \to y_i$ is a bijection for all $i < n$. This Γ works using Lemma 3.2. $\square$

We are now ready to prove Theorem 3.1. Suppose that κ^{+i+1} is collapsed in V_i where $i \in 2$ and $V_1 = V[G]$. Let μ be the cofinality of κ^{+i+1} in V_i. Fix a condition $\langle \vec{s}, \vec{F} \rangle$ as above. Suppose that $\dot{b}$ is a name for a function from μ cofinally into κ^{+i+1} such that $i_G(\dot{b}) \in V_i$. We may assume as in Lemma 2.9 that the forcing in the upper part of $\langle \vec{s}, \vec{F} \rangle$ is μ^+-closed. Note that for all Γ if Γ is a permutation of κ^+ fixing κ^{+i} then Γ fixes $\dot{b}$.

Applying the variation of the Prikry Lemma from the beginning of the section, we can obtain $\langle \vec{s}, \vec{G} \rangle \leq \langle \vec{s}, \vec{F} \rangle$ such that if $\langle \vec{t}, \vec{G}' \rangle \leq \langle \vec{s}, \vec{G} \rangle$ forces that $\dot{b}(\lambda) = \alpha$ then $\langle \vec{t}, \vec{G} \restriction \vec{t} \rangle$ forces $\dot{b}(\lambda) = \alpha$.

Define $A_\lambda = \{\alpha \mid \text{there is } \langle \vec{t}, \vec{G}' \rangle \leq \langle \vec{s}, \vec{G} \rangle \text{ such that } \langle \vec{t}, \vec{G}' \rangle \Vdash \dot{b}(\lambda) = \alpha\}$.

Claim 3.4. *$|A_\lambda| \leq \kappa^{+i}$.*

This claim is enough since the union of the A_λ has size $\leq \kappa^{+i}$ and therefore $\dot{b}$ cannot collapse κ^{+i+1}, a contradiction. Suppose the claim is false. Then $|A_\lambda| > \kappa^{+i}$. Then for each $\alpha \in A_\lambda$ there is $\langle \vec{t}_\alpha, \vec{F}^\alpha \rangle \leq \langle \vec{s}, \vec{G} \rangle$ which forces $\dot{b}(\lambda) = \alpha$. We may assume that $\vec{F}^\alpha = \vec{G} \restriction \vec{t}_\alpha$. First find a set unbounded in κ^{+i+1} such that all the conditions have the same length. Then if $i = 1$ there are only κ^+ many stems of a given length and it follows that there are α, β such that $\langle \vec{t}_\alpha, \vec{F}^\alpha \rangle = \langle \vec{t}_\beta, \vec{F}^\beta \rangle$, contradicting the choice of $\langle \vec{t}_\alpha, \vec{F}^\alpha \rangle$ and $\langle \vec{t}_\beta, \vec{F}^\beta \rangle$. If $i = 0$, then there are only κ many possibilities for the projected stems where we intersect each x with κ. It follows that there are α, β such that $\langle \vec{t}_\alpha, \vec{F}^\alpha \rangle$ and $\langle \vec{t}_\beta, \vec{F}^\beta \rangle$ satisfy the hypotheses of the Lemma 3.3. Hence there is an automorphism Γ such that $\Gamma(\langle \vec{t}_\alpha, \vec{F}^\alpha \rangle) \parallel \langle \vec{t}_\beta, \vec{F}^\beta \rangle$ but Γ fixes $\dot{b}$, again contradicting the choice of $\langle \vec{t}_\alpha, \vec{F}^\alpha \rangle$ and $\langle \vec{t}_\beta, \vec{F}^\beta \rangle$. This completes the proof of the claim and with it the proof of Theorem 3.1.

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