GITIK'S GAP 2 SHORT EXTENDER FORCING

SPENCER UNGER

1. Introduction

In this expository note, we seek to clarify work of Gitik [1]. In particular we give a proof of the following theorem.

Theorem 1.1. *Let κ be a singular cardinal of cofinality ω and $\langle \kappa_n \mid n < \omega \rangle$ be an increasing and cofinal sequence of cardinals where each κ_n is κ_n^{+n+2}-strong. There is a cardinal preserving forcing extension in which no bounded subsets of κ are added and $\kappa^\omega = \kappa^{++}$.*

In an Appalachian Set Theory Workshop [3], Gitik presented some of the details of a simplified version of the poset from his original paper. The discussion there motivates the definition of the forcing by modifying a poset which requires a stronger large cardinal assumption which is sometimes called the long extender forcing. However, the discussion recorded in the Appalachian Set Theory (AST) notes falls short of giving a complete account of this simplified forcing. The purpose of this paper is to fill in the gaps from the AST notes and so give a complete account of this simplified forcing. In particular we focus the proof that the final forcing has the κ^{++}-cc. Since the motivation for the poset is nicely outlined in the AST notes we jump straight into the technical details.

One note on convention is in order. We depart from previous presentations of this poset by using $p \leq q$ to mean that p is stronger than q.

2. Extenders

For the remainder of the paper we fix $j_n : V \to M_n$ witnessing that κ_n is κ_n^{+n+2}-strong and derive an extender $\langle E_{n\alpha} \mid \alpha < \kappa_n^{+n+2} \rangle$ where $E_{n\alpha} =_{def} \{X \subseteq \kappa \mid \alpha \in j(X)\}$.

Definition 2.1. *Let $n < \omega$. For $\alpha, \beta < \kappa_n^{+n+2}$, $\alpha \leq_{E_n} \beta$ if there is $f : \kappa_n \to \kappa_n$ such that $j_n(f)(\beta) = \alpha$.*

For $n < \omega$ and $\alpha < \beta < \kappa_n^{+n+2}$ with $\alpha \leq_{E_n} \beta$ we fix a witnessing projection $\pi_{\beta\alpha}$ as in the previous definition. These projections depend on n, but whenever we use them it will be clear which n is relevant. For use later we record three standard lemmas about extenders.

Lemma 2.2. *If $x \in [\kappa_n^{+n+2}]^{<\kappa_n}$, then there are unboundedly many $\alpha < \kappa_n^{+n+2}$ such that for all $\beta \in x$, $\beta \leq_{E_n} \alpha$.*

Date: September 14, 2014.

Lemma 2.3. *Let $\gamma < \beta \leq \alpha < \kappa_n^{+n+2}$. If $\gamma \leq_{E_n} \beta$ and $\beta \leq_{E_n} \alpha$, then the set $\{\nu < \kappa_n \mid \pi_{\alpha\beta}(\nu) > \pi_{\alpha\gamma}(\nu)\} \in E_{n\alpha}$. In particular for a set $x \in [\kappa_n^{+n+2}]^{<\kappa_n}$ with a $\leq_{E_n}$-maximal element α, there is a measure one set $A \in E_{n\alpha}$ satisfying the conclusion of the lemma for any $\gamma, \beta \in x$.*

Lemma 2.4. *Let $x \in [\kappa_n^{+n+2}]^{<\kappa_n}$ with a $\leq_{E_n}$-maximal element α. There is a set $A \in E_{n\alpha}$ such that for all $\gamma < \beta$ from x with $\gamma \leq_{E_n} \beta$ and all $\nu \in A$, $\pi_{\alpha\gamma}(\nu) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(\nu))$.*

3. Coding

We also need a notion of coding subsets of κ_n^{+n+2} of size less than κ_n for each n. So using GCH we fix a bijection $\mathrm{Code}_n : \kappa_n^{+n+2} \to [\kappa_n^{+n+2}]^{<\kappa_n}$ such that for all ordinals γ of cofinality at least κ_n^+, the restriction of Code_n to γ is a bijection from γ to $[\gamma]^{<\kappa_n}$. This coding is different from the one used by Gitik, but will serve the same purpose.

4. Good ordinals

A natural modification of the long extender forcing to the setting of short extenders collapses κ^{++}. The key technical difficulty is revise the forcing so that it has the κ^{++}-cc. For this Gitik uses the notion of a good ordinal.

For each $n < \omega$ and $k \leq n$ we consider the structure

$$\mathcal{A}_{n,k} = \langle H(\lambda^{+k}), \in, <, \lambda, E_n, \langle 0, 1, \dots \tau \cdots \mid \tau \leq \kappa_n^{+k} \rangle \rangle$$

where λ is a large regular cardinal, $<$ is a well-order of $H(\lambda^{+k})$ and each other parameter is a constant with the obvious interpretation. We write $\mathrm{tp}_{n,k}(\beta)$ for the collection of formulas in the language of $\mathcal{A}_{n,k}$ which β satisfies in $\mathcal{A}_{n,k}$. When n is implicit we refer to this type as 'the k-type of β.' For $\gamma < \lambda$, we expand $\mathcal{A}_{n,k}$ to $\mathcal{A}_{n,k}^\gamma$ by adding a constant symbol and interpreting it as γ. We let $\mathrm{tp}_{n,k}(\gamma, \beta)$ be the collection of formulas in the language of $\mathcal{A}_{n,k}^\gamma$ which are satisfied by β in $\mathcal{A}_{n,k}^\gamma$.

Remark 4.1. *A key feature of the above models is that for $k < k' \leq n$, $\mathcal{A}_{n,k} \in \mathcal{A}_{n,k'}$ using GCH. This will allow the model $\mathcal{A}_{n,k'}$ to make correct statements about the realization of types in $\mathcal{A}_{n,k}$. Moreover, by coding types as ordinals we may assume that there is a constant for each $\mathcal{A}_{n,k}^\gamma$-type in the structure $\mathcal{A}_{n,k}$.*

Remark 4.2. *Another key feature is that by GCH there are atmost κ_n^{++} many measures on κ_n. In particular, by coding these measures as ordinals we may assume that the structure $\mathcal{A}_{n,k}$ has a constant for each measure.*

Lemma 4.3 (Lemma 2.3 of [1]). *Let $n < \omega$. The set $\{\beta < \kappa_n^{+n+2} \mid \forall \gamma < \beta\ \mathrm{tp}_{n,n}(\gamma, \beta)$ is realized stationarily often below $\kappa_n^{+n+2}\}$ contains a club.*

Definition 4.4. *Let $k \leq n < \omega$ and $\beta < \kappa_n^{+n+2}$. We say that β is* k-good *if $\mathrm{cf}(\beta) \geq \kappa_n^{++}$ and for all $\gamma < \beta$, $\mathrm{tp}_{n,k}(\gamma, \beta)$ is realized stationarily often below κ_n^{+n+2}.*

Note that the previous lemma shows that there are many good ordinals. Good ordinals will be used to introduce some indiscernibility required for the chain condition argument. The indiscernibility required is captured by the next two lemmas.

Lemma 4.5 (Lemma 2.4 of [1])**.** *Let* $0 < k, l \leq n$*,* $\gamma < \rho < \kappa_n^{+n+2}$ *and* t *be a type in the language of* $\mathcal{A}^\gamma_{n,l}$ *realized by an ordinal above* γ*. If* $\mathrm{tp}_{n,k}(\gamma, \rho)$ *is realized stationarily often below* κ_n^{+n+2}*, then there is* η *with* $\gamma < \eta < \rho$ *such that* η *realizes* $t \restriction \min(k-1, l)$ *in* $\mathcal{A}^\gamma_{n,\min(k-1,l)}$*.*

Lemma 4.6 (Lemma 2.0 of [1])**.** *Let* $k \leq n < \omega$*. If* $\zeta, \zeta' < \kappa_n^{+n+2}$ *realize the same type in* $\mathcal{A}^\gamma_{n,k}$ *for some* $\gamma < \min(\zeta, \zeta')$*, then for every* ξ *with* $\zeta \leq \xi < \kappa_n^{+n+2}$ *there is a* ξ' *with* $\zeta' \leq \xi' < \kappa_n^{+n+2}$ *such that* $\mathrm{tp}_{n,k-1}(\zeta, \xi) = \mathrm{tp}_{n,k-1}(\zeta', \xi')$*.*

The proofs of both lemmas use Remark 4.1 in a crucial way.

Remark 4.7. *We can assume that the function* Code_n *is definable in the structure* $\mathcal{A}_{n,k}$ *by taking the* $<$*-least such coding function.*

Remark 4.8. Code_n *restricted to a good ordinal* γ *is a bijection from* γ *to* $[\gamma]^{<\kappa_n}$*.*

5. The poset

In this section we give the definition of the main forcing with all of the details. We begin with the definition of the cells Q_{n0} and Q_{n1} for $n < \omega$.

Definition 5.1. *Let* $Q_{n1} = \{f \mid f$ *is a partial function from* κ^{++} *to* κ_n *with* $|f| \leq \kappa\}$ *ordered by reverse inclusion.*

Definition 5.2. *Let* $(a, A, f) \in Q_{n0}$ *if*

(1) $f \in Q_{n1}$*,*
(2) a *is a partial order preserving function from* κ^{++} *to* κ_n^{+n+2} *with* $|a| < \kappa_n$ *such that*
  (a) $\mathrm{dom}(a)$ *has a largest element,*
  (b) $\mathrm{dom}(a) \cap \mathrm{dom}(f) = \emptyset$*,*
  (c) *for all* $\beta \in \mathrm{rng}(a)$*,* $\beta \leq_{E_n} \mathrm{mc}(a)$ *where* $\mathrm{mc}(a) = a(\max(\mathrm{dom}(a)))$ *and*
  (d) *for all* $\beta \in \mathrm{rng}(a)$*,* β *is atleast* 2*-good,*
(3) $A \in E_{n\,\mathrm{mc}(a)}$ *and*
(4) A *is contained in the sets given by Lemmas 2.3 and 2.4 applied to* $\mathrm{rng}(a)$*.*

Define $(a, A, f) \leq (b, B, g)$ *if*

(1) $f \leq g$ *in* Q_{n1}*,*
(2) $b \subseteq a$ *and*
(3) $\pi_{\mathrm{mc}(a)\,\mathrm{mc}(b)}$"$A \subseteq B$*.*

We are now ready to define the main poset with a Prikry type ordering.

Definition 5.3. *Let* $p = \langle p_n \mid n < \omega \rangle$ *be in* P *if there is* $l = \mathrm{lh}(p)$ *such that for all* $n < l$*,* $p_n \in Q_{n1}$ *and all* $n \geq l$*,* $p_n \in Q_{n0}$ *where if we write* $p_n = (a_n, A_n, f_n)$*, then the following properties hold:*

(1) *if* $m \geq n \geq l$*, then* $\mathrm{dom}(a_m) \supseteq \mathrm{dom}(a_n)$ *and*
(2) *if* $\alpha \in \mathrm{dom}(a_n)$ *for some* n*, then there is a nondecreasing sequence* $\langle k_m \mid n \leq m < \omega \rangle$ *such that* $k_m \to \infty$ *as* $m \to \infty$ *and for all* $m \geq n$*,* $a_m(\alpha)$ *is* k_m*-good.*

For conditions $p = \langle p_n \mid n < \omega \rangle$ *we adopt the convention that* $p_n = f^p_n$ *for* $n < \mathrm{lh}(p)$ *and* $p_n = (a^p_n, A^p_n, f^p_n)$ *for* $n \geq \mathrm{lh}(p)$*. For the ordering, we let* $p \leq q$ *if*

(1) $\mathrm{lh}(p) \geq \mathrm{lh}(q)$*,*
(2) *for all* $n < \mathrm{lh}(q)$*,* $f^p_n \leq f^q_n$ *in* Q_{n1}*,*

(3) *for all* n *with* $\mathrm{lh}(q) \leq n < \mathrm{lh}(p)$, *there is* $\nu \in A^q_n$ *such that* $f^p_n \leq f^q_n \cup \{(\xi, \pi_{\mathrm{mc}(a^q_n)a^q_n(\xi)}(\nu)) \mid \xi \in \mathrm{dom}(a^q_n)\}$ *and*
(4) *for* $n \geq \mathrm{lh}(p)$, $(a^p_n, A^p_n, f^p_n) \leq (a^q_n, A^q_n, f^q_n)$ *in* Q_{n0}.

For notational convenience we denote the minimal extension of a condition p *by a sequence* $\vec{\nu}$ *by* $p \frown \vec{\nu}$ *where* $\vec{\nu}$ *is a sequence of ordinals chosen from relevant measure one sets. Finally we define direct extension by* $p \leq^* q$ *if* $\mathrm{lh}(p) = \mathrm{lh}(q)$ *and* $p \leq q$.

This poset is not quite the poset that we want, but there is a projection from it to the final forcing. The reason for isolating this poset with the 'wrong' order is that it is of Prikry-type and we can prove the Prikry lemma.

Lemma 5.4. *If* D *is a dense open set in* P *and* $p \in P$, *then there are a* $q \leq^* p$ *and* $n < \omega$ *such that for all* $\langle \nu_0, \dots \nu_{n-1} \rangle \in \prod_{\mathrm{lh}(q) \leq i < \mathrm{lh}(q)+n} A^q_i$, $q \frown \vec{\nu} \in D$.

The proof is a straighforward adaptation of a similar proof for the long extender forcing. For a careful proof of this lemma for the long extender forcing we refer the reader to Lemma 2.18 of [2]. The Prikry lemma has the following corollaries.

Corollary 5.5. *Forcing with* $(P, \leq)$ *does not add any bounded subsets of* κ.

Corollary 5.6. *Forcing with* $(P, \leq)$ *preserves* κ^+.

Next we seek to define a projection of $(P, \leq)$ which will be the final forcing.

Definition 5.7. *Let* $1 < k \leq n$ *and* $(a, A, f), (b, B, g) \in Q_{n0}$. *We define* $(a, A, f) \leftrightarrow_{n,k} (b, B, g)$ *if*
(1) $f = g$,
(2) $A = B$,
(3) $\mathrm{dom}(a) = \mathrm{dom}(b)$ *and*
(4) $\mathrm{Code}_n^{-1}(\mathrm{rng}(a))$ *and* $\mathrm{Code}_n^{-1}(\mathrm{rng}(b))$ *realize the same* k*-type.*

It is not obvious that we can satisfy clause (2). It could be that $\mathrm{mc}(a) \neq \mathrm{mc}(b)$. In which case it is not obvious that we can have $A = B$, since perhaps $E_{n\,\mathrm{mc}(a)} \neq E_{n\,\mathrm{mc}(b)}$. However by Remark 4.2, clause (4) implies that the two measures are equal, since we can express "$E_{n\,\mathrm{mc}(a)}$ is the δ^{th} measure in some enumeration of measures on κ_n" as a sentence about $\mathrm{Code}_n^{-1}(\mathrm{rng}(a))$ in $\mathcal{A}_{n,k}$.

Definition 5.8. *Let* $p, q \in P$. *We define* $p \leftrightarrow q$ *if*
(1) $\mathrm{lh}(p) = \mathrm{lh}(q)$,
(2) *for all* $n < \mathrm{lh}(p)$, $p_n = q_n$ *and*
(3) *there is a nondecreasing sequence* $\langle k_m \mid \mathrm{lh}(p) \leq m < \omega \rangle$ *such that* $k_m \to \infty$ *as* $m \to \infty$ *and for all* $m \geq \mathrm{lh}(p)$, $p_m \leftrightarrow_{m,k_m} q_m$.

Using this notion we can define the restricted ordering on P.

Definition 5.9. *Let* $p, q \in P$. *We define* $p \to q$ *if there is a sequence of elements* $\langle r_i \mid i < m \rangle$ *such that* $r_0 = p$, $r_{m-1} = q$ *and for all* $i < m$ *either* $r_{i+1} \leq r_i$ *or* $r_{i+1} \leftrightarrow r_i$.

Note that $p \to q$ means q is stronger than p, which is the same as the convention used by Gitik. The proofs of the following facts are straightforward.

Lemma 5.10. *The identity map is a projection from* $(P, \leq)$ *to* $(P, \to)$.

Corollary 5.11. *Forcing with* $(P, \to)$ *preserves cardinals below* κ^{++}.

Lemma 5.12. *$(P, \to)$ adds κ^{++} cofinal ω-sequences in κ.*

So to finish the proof it is enough to prove that κ^{++} is preserved.

Lemma 5.13. *$(P, \to)$ has the κ^{++}-cc*

We break the argument into two claims. First if two conditions are similar enough then they can be extended to equivalent conditions in the sense of Definition 5.8. Second a Δ-system argument to show that a κ^{++} sequence of conditions can be refined to have many similar pairs of conditions. For ease of notation we define $X_p = \bigcup_{n \geq \text{lh}(p)} (\text{dom}(a_n^p) \cup \text{dom}(f_n^p))$.

Lemma 5.14. *If $p, q \in P$ with the following properties*

(1) $\text{lh}(p) = \text{lh}(q)$,
(2) *for $n < \omega$, f_n^p is compatible with f_n^q and*
(3) *there is an ordinal β such that*
  (a) $\beta \cap X_p = \beta \cap X_q$,
  (b) *for all $n \geq \text{lh}(p)$, $a_n^p \restriction (X_p \cap \beta) = a_n^q \restriction (X_p \cap \beta)$,*
  (c) *every element of $X_p \setminus \beta$ is above every element of $X_q \setminus \beta$,*
  (d) *for all $n \geq \text{lh}(p)$, $\text{otp}(\text{dom}(a_n^p) \setminus \beta) = \text{otp}(\text{dom}(a_n^q) \setminus \beta)$ and if δ is this common ordertype then for all $i < \delta$, $a_n^p(\alpha) = a_n^q(\alpha')$ where α and α' are the i^{th} elements of $\text{dom}(a_n^p) \setminus \beta$ and $\text{dom}(a_n^q) \setminus \beta$ respectively and*
  (e) *If $\rho_p = \min(X_p \setminus \beta)$ and $\rho_q = \min(X_q \setminus \beta)$, then $\rho_p \in \text{dom}(a_{\text{lh}(p)}^p)$, $\rho_q \in \text{dom}(a_{\text{lh}(q)}^q)$ and $a_{\text{lh}(p)}^p(\rho_p) = a_{\text{lh}(q)}^q(\rho_q)$ is atleast 5-good,*

then there are conditions $p' \leq p$ and $q' \leq q$ such that $p' \leftrightarrow q'$.

Proof. The constructions of p' and q' are different so we take them each in turn. Before we begin the construction, we let $\langle k_m \mid \text{lh}(p) \leq m < \omega \rangle$ witness clause (2) for ρ_p and ρ_q in Definition 5.3. We also choose β^* greater than all ordinals appearing in the domains of a_n^p, f_n^p, a_n^q and f_n^q for $n < \omega$.

First we construct p'. By assumption for all $n < \omega$, $f_n^{p'} = f_n^p \cup f_n^q$ is a condition in Q_{n1}. The issue is to add $\text{dom}(a_n^q) \setminus \beta$ to the domain of $\text{dom}(a_n^p)$ to obtain $a_n^{p'}$. To do this we let $n \geq \text{lh}(p)$ and $k = k_n$ and apply Lemma 4.5 with $\gamma = \text{Code}_n^{-1}(\text{rng}(a_n^p \restriction \beta))$, $\rho = a_{\text{lh}(p)}^p(\rho_p)$, $t = \text{tp}_{n,k-1}(\text{rng}(a_n^p \restriction (\text{dom}(a_n^p) \setminus \beta)))$ and $l = k - 1$. The output of the lemma is an ordinal η such that $\text{Code}_n(\eta)$ is strictly between a_n^p"$(\text{dom}(a_n^p) \cap \beta)$ and $a_{\text{lh}(p)}^p(\rho_p)$ and has the same ordertype as a_n^p"$(\text{dom}(a_n^p) \setminus \beta)$. If δ is this common order type, $i < \delta$ and the i^{th} element of a_n^p"$(\text{dom}(a_n^p) \setminus \beta)$ is k'-good for some k', then the i^{th} element of $\text{Code}_n(\eta)$ is $\min(k - 1, k')$-good. It is not hard to see now that we can obtain an order preserving function $a_n^{p'}$ by adding $\text{dom}(a_n^q) \setminus \beta$ to the domain of a_n^p and mapping its i^{th} element to the i^{th} element of $\text{Code}_n(\eta)$. To make $a_n^{p'}$ a candidate for being a condition we add β^* to the domain and map it to an $n-2$-good ordinal which becomes the maximal (in both the ordinal sense and in $\leq_{E_n}$) element of $\text{rng}(a_n^{p'})$. We leave off the construction of $A_n^{p'}$ until the end.

Next we construct q'. Again we set $f_n^{q'} = f_n^p \cup f_n^q$ for all $n < \omega$. This time the issue is to add $\text{dom}(a_n^p) \setminus \beta$ to $\text{dom}(a_n^q)$ to obtain $a_n^{q'}$. To do this we let $n \geq \text{lh}(q)$ and $k = k_n$. Now we apply Lemma 4.6 with

- $k - 1$ in place of k,
- $\gamma = 0$,
- $\zeta = \text{Code}_n^{-1}(\text{rng}(a_n^p \restriction \beta) \cup \text{Code}_n(\eta))$ (for η as above),

- $\zeta' = \mathrm{Code}_n^{-1}(\mathrm{rng}(a_n^p))$ and
- $\xi = \mathrm{Code}_n^{-1}(\{a_n^{p'}(\beta^*)\} \cup \mathrm{rng}(a_n^q \restriction \mathrm{dom}(a_n^q) \setminus \beta))$.

For the hypothesis of Lemma 4.6, we need to check that ζ and ζ' realize the same k-type. This follows from the fact that $\mathrm{Code}_n(\eta)$ and $\mathrm{rng}(a_n^p \restriction (\mathrm{dom}(a_n^p) \setminus \beta))$ realize the same $k-1$-type over $\mathrm{Code}_n^{-1}(\mathrm{rng}(a_n^p \restriction \beta))$ which comes from our application of Lemma 4.5 above. The conclusion of Lemma 4.6 gives an ordinal ξ' realizing the same $k-2$-type over ζ' as ξ does over ζ. Similar to before we add $\{\beta^*\} \cup \mathrm{dom}(a_n^p) \setminus \beta$ to $\mathrm{dom}(a_n^q)$ and map it to $\mathrm{Code}_n(\xi')$ in the natural way. Note that the maximum (in the ordinal sense) element of $\mathrm{Code}_n(\xi')$ is $\leq_{E_n}$-above every element of $\mathrm{rng}(a_n^{q'})$, since $a_n^{p'}(\beta^*)$ is $\leq_{E_n}$-above every element of $\mathrm{rng}(a_n^{p'})$ and this is expressable as a formula in $\mathrm{tp}_{n,k-2}(\zeta, \xi)$. It follows that $\mathrm{rng}(a_n^{p'})$ and $\mathrm{rng}(a_n^{q'})$ realize the same $k-2$-type.

Finally, we define $A_n^{p'} = A_n^{q'}$ using the fact that $E_{n\,\mathrm{mc}(a_n^{p'})} = E_{n\,\mathrm{mc}(a_n^{q'})}$ from the remarks following Definition 5.7. We choose this common measure one set so that its projections via $\pi_{\mathrm{mc}(a_n^{p'})\,\mathrm{mc}(a_n^p)}$ and $\pi_{\mathrm{mc}(a_n^{q'})\,\mathrm{mc}(a_n^q)}$ are contained in A_n^p and A_n^q respectively. It is not hard to see that both p' and q' are conditions and that $p' \leftrightarrow q'$ as witnessed by the sequence $\langle k_m - 2 \mid m \geq \mathrm{lh}(p) \rangle$. □

We are left with proving the following lemma.

Lemma 5.15. *Given $\langle p_\alpha \mid \alpha < \kappa^{++} \rangle$ there is a set $I \subseteq \kappa^{++}$ of size κ^{++} such that for all α, β in I, p_α and p_β satisfy the hypotheses of the previous lemma.*

Proof. For ease of notation we let $a_n^{p_\alpha} = a_n^\alpha$ for $n < \omega$ and $\alpha < \kappa^{++}$ and similarly for $f_n^{p_\alpha}$. The set I is a result of many refinements, but for ease of notation we just reindex after each step.

We begin by fixing the length of the conditions on a set of size κ^{++}. Let l be the common length. Next we form a Δ-system out of $\{\bigcup_{n \geq l}(\mathrm{dom}(f_n^\alpha) \cup \mathrm{dom}(a_n^\alpha)) \mid \alpha < \kappa^{++}\}$. The typical pressing down phase of the Δ-system argument gives the ordinal β required with a little more work.

For $n \geq l$ we can assume that the following are fixed

1. the root r of the delta system,
2. $a_n^\alpha \restriction r$ and $f_n^\alpha \restriction r$,
3. the ordertype of $\mathrm{dom}(a_n^\alpha) \setminus r$,
4. a_n^α“$(\mathrm{dom}(a_n^\alpha) \setminus r)$,
5. the sequence $\langle k_m \mid m \geq l \rangle$ witnessing clause (2) of Definition 5.3 for $a_n^\alpha(\rho_\alpha)$ where $\rho_\alpha = \min(\bigcup_{n \geq l}(\mathrm{dom}(a_n^\alpha) \setminus r))$ and
6. the minimum k such that $\rho_\alpha \in \mathrm{dom}(a_k^\alpha)$.

Let $l^* \geq k$ be large enough so that $k_{l^*} \geq 5$. Extend each condition in a minimal way to have length l^*. Finally an easy Δ-system argument involving the conditions f_n^α for $n < l^*$ finishes the proof. □

Remark 5.16. *We actually proved that $(P, \rightarrow)$ has the κ^{++}-Knaster property.*

References

1. Moti Gitik, *Blowing up the power of a singular cardinal*, Annals of Pure and Applied Logic **80** (1996), no. 1, 17 – 33.
2. ———, *Prikry-type forcings*, Handbook of Set Theory (Matthew Foreman and Akihiro Kanamori, eds.), Springer Netherlands, 2010, pp. 1351–1447 (English).

3. Moti Gitik and Spencer Unger, *Short extender forcing*, Appalachian Set Theory 2006-2012 (Ernest Schimmerling and James Cummings, eds.), Cambridge University Press, 2012.