

TREES IN SET THEORY

SPENCER UNGER

1. INTRODUCTION

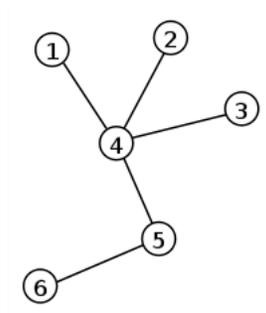
Although I was unaware of it while writing the first two talks, my talks in the graduate student seminar have formed a coherent series. This talk can be viewed as the third in series entitled ‘The shortest path to difficult topics in set theory.’ The first talk that I gave was on the subject of forcing. The second talk was on the subject of large cardinals. This talk will be on trees and their place in set theory. The series is going to be culminated in this talk, since in connection with trees I get to mention both forcing and large cardinals.

2. MATHEMATICAL BACKGROUND

I thought that I would approach the notion of tree in set theory from the notion of tree in combinatorics, since that is probably more familiar. Recall that a graph is a set of vertices V and a set of edges between them E , which is best viewed as a subset of $[V]^2$, the set of unordered pairs of vertices.

Definition 2.1. *A graph is a tree if it is connected and has no cycles.*

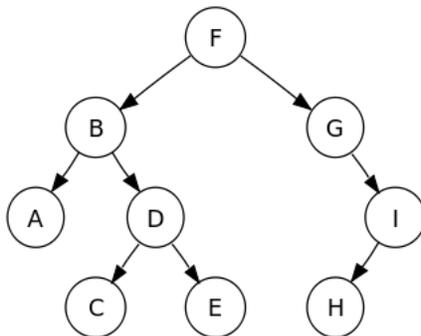
A typical picture (taken from Wikipedia).



Now suppose that we choose a distinguished vertex, which we’ll call the root. This allows us to turn our graph in to a directed graph in a natural way. We simply say that vertex u is below vertex v if it is closer to the root. Since our tree is connected and acyclic, this makes sense.

Here is a typical picture of a rooted tree with its directed graph structure (Taken from a Wikibook on programming).

Date: November 21, 2012.



So here the root is the vertex labelled ‘F’. Since we now have a sensible direction for each edge, this defines a relation on the set V . Notice that it makes sense to take the transitive closure of this relation to produce an ordering. Orderings like this are *partial*, because for example the nodes ‘A’ and ‘D’ are *incomparable* in the ordering. The properties of orderings that arise in this way can be axiomatized as follows.

Defintion 2.2. A finite-ish, set theoretic tree is a set of nodes T together with a binary relation $<_T$ on T such that

- (1) $<_T$ is transitive and irreflexive,
- (2) for every $t \in T$, the set $\{u \in T \mid u <_T t\}$ is finite and linearly ordered by $<_T$, and
- (3) for every $t, u \in T$, there is $v \in T$ with $v \leq_T t, u$.¹

So we have essentially proved one direction of the following easy proposition.

Proposition 2.3. Every graph theoretic tree with a chosen root corresponds in a natural way to a finite-ish set theoretic tree and vice versa.

So there is a bit of a catch. And that is with the word ‘finite’ in the definition of a finite-ish set theoretic tree. When we remove the word finite from condition (2) in the definition, these trees no longer arise as the ‘rooting’ of a graph theoretic tree. This is because of the essentially local nature of the edge relation in a graph and the rather complex notion of a transitive ordering. Consider the natural ordering on the set $\mathbb{N} \cup \{\infty\}$. This satisfies all of the set theoretic tree definitions except for the finite part, but it does not arise as a the rooting of a graph theoretic tree. In particular ∞ would have to be connected to some element of \mathbb{N} , which kills the fact that it is supposed to be above all elements of \mathbb{N} in the tree ordering.

Here is the full definition of a set theoretic tree (which from here on we’ll call just call a tree). Recall that a transitive relation is wellfounded if it has no infinite decreasing sequences. As we’ll see, this additional requirement imposes alot of nice structure.

Defintion 2.4. $(T, <_T)$ is a tree if

- (1) $<_T$ is a wellfounded partial ordering on T ,
- (2) for all $t \in T$, $\{u \in T \mid u <_T t\}$ is linearly ordered by $<_T$ and
- (3) for all $t, u \in T$, there is $v \in T$ such that $v \leq_T t$ and $v \leq_T u$.

¹Here $t \leq_T t'$ literally means $t <_T t'$ or $t = t'$.

3. TREES IN SET THEORY

We'll start out by giving some examples. The following are all trees.

- $\mathbb{N} \cup \{\infty\}$ ordered in the natural way.
- The set of finite binary sequences ordered by end extension.
- The set of finite and countably infinite sequences of natural numbers ordered by end extension.

Here is a natural definition.

Definition 3.1. *A branch b through a tree T is a subset of the tree linearly ordered by $<_T$.*

So for example we can view infinite binary sequences as branches through the tree of finite binary sequences. Literally if s is an infinite binary sequence, then the set $\{s \upharpoonright n \mid n \in \mathbb{N}\}$ is a branch through the tree.

The first theorem that one learns about in the theory of trees is König infinity lemma. A tree is *finitely branching* if each node has finitely many immediate successors.

Theorem 3.2. *Every countably infinite, finitely branching tree has an infinite branch.*

Proof. Construct the branch by induction. Start at the root. Since the tree is finitely branching, choose an immediate successor with infinitely many nodes above it. Then continue picking in the same way. The result is an infinite branch. \square

To go further we need a digression in to set theory on the topic of ordinals and cardinals.

4. ORDINALS AND CARDINALS

A few definitions.

Definition 4.1. *A set X is transitive if and only if for every x, y , if $x \in y \in X$, then $x \in X$.*

We are interested in transitive sets, because we are going to turn \in into an ordering on our sets. To make sense of this we need our sets to be transitive.

Definition 4.2. *$(W, <)$ is a well ordering if and only if it is a linear ordering with no infinite decreasing chain.*

Set theorists like things with 'well' in front of them. We'll see why in a second.

Definition 4.3. *A set α is an ordinal if and only if α is transitive and (α, \in) is a well order.*

Some facts about ordinals.

Fact 4.4. *If α is an ordinal, then $\alpha \cup \{\alpha\}$ is an ordinal. The union of a set of ordinals is an ordinal.*

This gives a more constructive way to think about the ordinals. To get from an ordinal α to its successor in the \in ordering, I take $\alpha \cup \{\alpha\}$. If at some point I reach an infinite limit of ordinals, then I can just take the union of the sets that I've built so far.

Fact 4.5. *Every well ordering is isomorphic to an ordinal.*

This fact is really important. It shows that the ordinals characterize all well orderings. So in some sense they are the only ones that we need to consider. If $(W, <)$ is a well-ordering, then we'll denote the unique ordinal, which it is isomorphic to by $\text{o.t.}(W, <)$. We'll drop the ordering if it is natural to do so. Some ordinals are special in the following sense.

Defintion 4.6. *An ordinal κ is a cardinal if and only if there is no surjection from an ordinal $\beta < \kappa$ on to κ .*

Cardinals that you know:

- $\text{o.t.}(\{0, \dots, n-1\})$
- $\text{o.t.}(\mathbb{N})$
- If $(\mathbb{R}, <_{\mathbb{R}})$ is a well ordering of the reals, then $\text{o.t.}(\mathbb{R}, <_{\mathbb{R}})$ is a cardinal.

Set theorists call these n , \aleph_0 and 2^{\aleph_0} respectively.

Cardinals you might not know: for all $n \geq 1$, \aleph_n is the n^{th} uncountable cardinal. In particular \aleph_1 is the least uncountable cardinal. The Continuum Hypothesis is the statement that the size of the reals is the least uncountable cardinal.

Fact 4.7. *(Assuming the axiom of choice) Every set is in bijection with a unique cardinal.*

5. DEFINITIONS ABOUT TREES WITH ORDINALS

Using ordinals we can define many natural objects.

Defintion 5.1. *Assume that $(T, <_T)$ is a tree.*

- For $t \in T$, the height of t is $\text{ht}(t) = \text{o.t.}\{u \in T \mid u <_T t\}$.
- The α^{th} level of T is $\text{Lev}_{\alpha}(T) = \{t \in T \mid \text{ht}(t) = \alpha\}$.
- The height of the tree $\text{ht}(T)$ is the least α so that $\text{Lev}_{\alpha}(T) =$

6. THE TREE PROPERTY

With this we can formulate a generalization of König Infinity Lemma.

Defintion 6.1. *Let κ be a cardinal. T is a κ -tree if $\text{ht}(T) = \kappa$ and every level has size less than κ .*

Defintion 6.2. *A κ -tree is Aronszajn if it has no branches of order type κ .*

Recall that a branch is just a subset of the tree linearly ordered by the tree relation.

Defintion 6.3. *A cardinal κ has the tree property if there are no κ -Aronszajn trees. (Equivalently we could have said 'every κ -tree has a branch of ordertype κ .)*

Note that König infinity lemma is precisely the statement that \aleph_0 has the tree property.

Theorem 6.4. *There is an \aleph_1 -Aronszajn tree.*

Recall that \aleph_1 is the least uncountable cardinal. So this result says that \aleph_1 does not have the tree property.

The first tree that we'll construct is a tree of increasing sequences of rational numbers with a top element. As a corollary of the proof we'll have the following fact.

Fact 6.5. *Let α be a countable ordinal. There is $A \subseteq \mathbb{Q}$ such that A with the usual ordering on the rationals is isomorphic to α .*

Before the construction a small piece of notation. If t is an increasing sequence of rationals with a top element, then $\text{top}(t)$ denotes the top element of t . We'll construct by induction on ordinals $\alpha < \aleph_1$, a tree T of increasing sequences of rationals with a top element satisfying the following properties.

- (1) Suppose that $\alpha < \beta < \aleph_1$, $t \in \text{Lev}_\alpha(T)$ and $r \in \mathbb{Q}$ with $r > \text{top}(t)$, then there is a $u \in \text{Lev}_\beta(T)$ such that $t \leq u$ and $r > \text{top}(u)$.
- (2) For all $\alpha < \aleph_1$, $\text{Lev}_\alpha(T)$ is countable.

So induction on ordinals is just like induction on \mathbb{N} except that we have to deal with 'limit stages'. Suppose that we've constructed $\text{Lev}_n(T)$ for all $n \in \mathbb{N}$, then we need a way to continue the construction, that is put points on top of some of the branches that we've constructed so far.

We need some kind of base case, so we let $\text{Lev}_0(T) = \{\langle 0 \rangle\}$.

Now the successor step. Suppose that we've constructed the tree up through level β . We have a tree we've constructed so far, which we call T_β . We want to end extend the sequences on level β to construct level $\beta + 1$. Moreover we want to do this while satisfying conditions (1) and (2) above.

Work on condition (1), suppose that $t \in \text{Lev}_\beta(T)$ and that $r > \text{top}(t)$, then by the density of the rationals we can pick a q with $\text{top}(t) < q < r$. We then add $t \frown \langle q \rangle$ to the tree. This completes the construction of the level $\beta + 1$. We only added countably many points (thus satisfying condition (2)), since $\text{Lev}_\beta(T)$ is countable by induction and \mathbb{Q} is countable.

Suppose that γ is a limit ordinal and we have constructed all levels for ordinals less than γ . So we have a tree which we'll call $T_{<\gamma}$. We need to put points on top some of the branches through $T_{<\gamma}$ in a way that satisfies (1) and (2).

Since γ is countable and it is not equal to $\alpha \cup \{\alpha\}$ for any α . We can choose an increasing sequence of ordinals α_n for $n \in \mathbb{N}$, which is cofinal in γ . By this I mean that for all $\beta < \gamma$ there is an n such that $\beta < \alpha_n$.

We want satisfy (1). So we take $t \in T_{<\gamma}$ and $r \in \mathbb{Q}$ above the top of t . Then we construct a branch through $T_{<\gamma}$ that extends t and stays below r . Let $n(t)$ be the least n such that $ht(t) < \alpha_n$.

We construct t_n for $n \geq n^t$ with the following properties,

- $t = t_{n(t)}$,
- for all $n \geq n(t)$, $t_n \in \text{Lev}_{\alpha_n}(T)$
- for all $n \geq n(t)$, t_{n+1} end extends t_n .

We are just going to apply the inductive hypothesis (2) countably many times. Fix in advance r' so that $\text{top}(t) < r' < r$. Suppose that we've constructed up to stage n . Then apply the inductive hypothesis to with levels $\alpha_n < \alpha_{n+1}$, t_n and r' to obtain t_{n+1} . Then add $\cup_{n \geq n(t)} t_n \frown \langle r' \rangle$ to the tree.

Similar counting to before shows that we still have (2).

This does it. Notice that our tree T cannot have a branch of order type \aleph_1 , since that would be an uncountable strictly increasing sequence of rationals. More over the function top is a function from T to \mathbb{Q} such that if t end extends t' , then $\text{top}(t) \neq \text{top}(t')$. Such a function is called a *specializing function*. It provides a concrete witness that the tree has no branches.

It is natural to ask about ω_2 . The ‘answer’ here is much more complex, since it involves the notions of consistency and large cardinals. To introduce this let me say a few things about consistency and models of set theory.

Informal Definition 6.6. *A model of a set of axioms is a structure where those axioms are true*

So for example the reals with the usual operations are a model of the field axioms. The $n \times n$ matrices with coefficients in \mathbb{C} together with the operator norm are a model of the axioms of C^* -algebras. A model of set theory is a set M together with an interpretation of the \in^M relation, which satisfies the axioms of ZFC set theory.

Informal Definition 6.7. *A set of axioms is consistent if it has a model*

Defintion 6.8. *Let κ be an uncountable cardinal. κ is weakly compact if for every map $f : [\kappa]^2 \rightarrow 2$ there is an H of size κ such that f is constant on $[H]^2$.*

Notice that the ‘weak compactness’ of \aleph_0 is exactly infinite ramsey theorem. However, the existence of a weakly compact cardinal cannot be proven from the axioms of set theory. Much of modern set theory begins by assuming ZFC and the existence of large cardinals. Why? The answer is that there are statements in set theory whose strength can only be measured by large cardinals. Consider the following theorem.

Theorem 6.9. *The axioms ZFC + ‘there is a weakly compact cardinal’ are consistent if and only if the axioms ZFC + ‘ \aleph_2 has the tree property’ are consistent*

Let me say something about the proof. The reverse direction of the theorem requires the use of the Gödel’s constructible universe L . Using L Gödel proved that the continuum hypothesis is consistent with ZFC. The forward direction of the theorem requires the notion of forcing, which was developed by Paul Cohen to prove that the failure of the continuum hypothesis is consistent with ZFC. (See my Graduate Student Seminar on Forcing for more detail.) The idea of the construction is to start with a model of set theory and pass to a larger one in which the desired axioms hold. In particular we start with a model of ZFC with a weakly compact cardinal and we add objects which show that the some of the cardinals of the previous model are not actually cardinals and at the same time we add many elements of the powerset of the natural numbers. In the end, we have that the continuum is our formerly weakly compact cardinal is \aleph_2 .

Having said a bit extra about a the forcing direction of the theorem, I can also mention a theorem of mine.

Theorem 6.10. *Let α be an ordinal. If the axioms ZFC + ‘there is a weakly compact cardinal’ are consistent, then the axioms ZFC + ‘ \aleph_2 has the tree property’ + ‘the continuum is greater than α ’ are consistent.*

This is a strengthening of the forward direction of Mitchell’s theorem. That’s all. Thanks!