Logic Seminar
Basic PCF

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So for the last two talks we dug in to a very general construction that has a lot of bearing on the PCF. However for those of you who don't know much about the PCF, the application was unclear. So now I was thinking that I would go back and review the basic PCF definitions and then slowly work back to the importance of Dichotomy and Trichotomy in the theory. I'm not sure how far I will get in today, but there are always the weeks until Will gets his stuff together. I suppose that I could talk about something else, but PCF is just so cool!

I any case lets go over the basic definitions. We are going to work with functions modulo an ultrafilter, but unlike the general set up we are going to work with a certain set of functions. Before we do that, let me give some definitions that are interesting in their own right.

Definition. *Let $\mathbb{P}$ be a poset. The $A \subseteq \mathbb{P}$ is cofinal if and only if for all $b \in \mathbb{P}$, there is an $a \in A$ such that $a \geq b$. Then let* $\mathrm{cf}(\mathbb{P}) = \min\{|A| : A$ *is cofinal in* $\mathbb{P}\}$.

Remark. One can check that $\mathrm{cf}(\lambda) = \mathrm{cf}(\lambda, \leq)$.

Example. $\mathrm{cf}(\mathbb{P})$ might be singular! Take the disjoint union of the ω_ns. Then for each n there is an unbounded set of size ω_n that is not cofinal! So the confinality of this this poset is ω_ω.

Definition. *$\mathbb{P}$ has true cofinality if and only if it has a cofinal linearly ordered set.*

We adopt the convention that our posets $\mathbb{P}$ satisfy, for all $a \in \mathbb{P}$ there is a $b \in \mathbb{P}$ with $b > a$.

Fact. *If $\mathbb{P}$ has true cofinality, then* $\mathrm{cf}(\mathbb{P})$ *is a regular cardinal.*

1 Definition of $\mathrm{pcf}(A)$

Let A be a set of Regular cardinals.

Definition. *A is progressive if and only if $|A|^+ < \min(A)$.*

This is going to be an important definition for us. It avoids certain pathologies and generally makes things run more smoothly. Our canonical example will be $A_0 = \{\aleph_n : 2 \leq n < \omega\}$.

Let $\prod A = \{f : A \to \mathrm{On} : \forall\ a \in A\ f(a) < a\}$ We are then going to order these modulo an ultrafilter on A, U say. Then $\prod A/U$ is defined exactly like you would expect from the last couple of talks. If $f, g \in \prod A$, then $f <_U g$ iff $\{a \in A : f(a) < g(a)\} \in U$. And $\prod A/U$ is a linear order. Moreover, it is a linear order with an upperbound in ${}^A\mathrm{On}$, namely $h(a) = a$ for all $a \in A$. So it makes sense to ask about the cofinality of this ordering.

Moreover, we can deal with cofinality as introduced above. This allows us to define $\mathrm{pcf}(A)$.

Definition. $\mathrm{pcf}(A) = \{\mathrm{cf}(\prod A/U) : U$ *is an ultrafilter on* $A\}$.

Much work in PCF theory goes in to bounding the size of pcf. It turns out that $\mathrm{pcf}(A)$ is intimately tied to certain kinds of elementary substructures. And more over these substructures have connections with cardinal arithmetic. This is one of the ideas behind Shelah's cardinal arithmetic result that if $\aleph_\omega$ is string limit then $2^{\aleph_\omega} < \aleph_{\omega_4}$.

Niavely, we have $|\mathrm{pcf}(A)| \leq 2^{2^{|A|}}$, which comes from a bound on the number of possible ultrafilters. This bound is greatly improved through the study of PCF theory.

Definition. *An ultrafilter U on A is principal, if and only if there is an $a \in A$ such that $U = \{X \subseteq A : a \in X\}$.*

The principal ultrafilters represent sort of uninteresting points in $\mathrm{pcf}(A)$. Fix $a \in A$ and suppose that $U = \{X \subseteq A : a \in X\}$. Then $\mathrm{cf}(\prod A/U) = a$. Lets think about this. Suppose that we have $f, g \in \prod A$. Then to ask the question is $f <_U g$? Is really just asking the question is $f(a) < g(a)$? ie $f <_U g$ if and only if $f(a) < g(a)$.

So lets think about the cofinality. Define a cofinal sequence $g_i \in \prod A$ for $i < a$ by $g_i(a) = i$ for all $i < a$ and anything that you want otherwise. So the cofinality is less than or equal to a, but it can't be less than a, because any shorter sequence is bounded below a on the ath coordinate since a is regular.

So in general we have $A \subseteq \mathrm{pcf}(A)$!

Continuing our example from above (A_0 is the $\aleph_n$s), let F be the filter of cofinite sets. If U is a nonpricipal ultrafilter then $F \subseteq U$. Otherwise, there would be a finite set in U from which we could get a singleton in U contradicting that U is nonpricipal. We will show that the cofinality of the product of A_0 modulo such a U is in fact larger than $\aleph_\omega$.

Definition. *A poset $\mathbb{P}$ is μ-directed (for μ a cardinal) if and only if for all $A \in [\mathbb{P}]^{<\mu}$, there is $p \in \mathbb{P}$ such that for all $a \in A$, $a < p$.*

Proposition 1. *The product $\prod A_0/F$ is $\aleph_{\omega+1}$-directed, where A_0, F are from above.*

Proof. We are only worried about so-called eventual domination. That is $f <_F g$ if there is a N such that for all $n \geq N$, $f(n) < g(n)$. Let $B \subseteq \prod A_0$ with $|B| = \aleph_\omega$. Write B as $\bigcup_n B_n$ such that $B_N \subseteq B_{n+1}$ and $|B_n| = \aleph_n$. Define $g(n) = \sup_{h \in B_{n-1}} h(n)$. Note that such a supremum remains in the $\prod A_0$, becuase at each coordinate n we take a supremum of ω_{n-1} things. Then for all $h \in \prod A_0$, we have $h <_F g$, because each h appears in some B_N and this N witnesses the evntual domination of h by g. $\square$

So we conclude that $\prod A_0/U$ is $\aleph_{\omega+1}$-directed for a nonpricipal U. Otherwise, we would have a bounded set of size $\aleph_\omega$ which was cofinal, but this is impossible.

2 Facts about $J_{<\lambda}(A)$

Definition. *Let A be progressive and λ be a cardinal. Then $J_{<\lambda}(A) =_{def} \{B \subseteq A : \forall U$ a uf,$B \in U \Rightarrow \mathrm{cf}(\prod A/U) < \lambda\}$*

We will drop the parameter A where there is no risk of confusion. We will almost always work with A progressive.

Proposition 2. *$J_{<\lambda}(A)$ is an ideal on A. (Possibly improper).*

Proof. For no U is $\emptyset \in U$, so $\emptyset \in J_{<\lambda}(A)$. Given $B_1, B_2 \in J_{<\lambda}(A)$, let U be an ultrafilter with $B_1 \cup B_2 \in U$. Then $B_1 \in U$ or $B_2 \in U$ as U is an ultrafilter. In either case we have $\mathrm{cf}(\prod A/U) < \lambda$. Lastly, suppose $B_2 \in J_{<\lambda}(A)$ and $B_1 \subseteq B_2 \subseteq A$. Fix an ultrafilter with $B_1 \in U$. $B_1 \in U$ implies $B_2 \in U$. So $\mathrm{cf}(\prod A/U) < \lambda$. $\square$

The talk has been kind of a survey so far with some proofs and now I will change in to full survey mode. First, some general facts about $J_{<\lambda}$.

Theorem 1. *$\prod A/J_{<\lambda}$ is λ-directed*

Which helps us prove

Theorem 2. *For any ultrafilter U on A and any λ, $\mathrm{cf}(\prod A/U) < \lambda \Leftrightarrow U \cap J_{<\lambda}(A) \neq \emptyset$*

This is a nice characterization of the PCF in terms of the $J_{<\lambda}$s. The $J_{<\lambda}$s are essential to the study of PCF theory.

Using some basic theory about the $J_{<\lambda}$ we get

Theorem 3. $|\mathrm{pcf}(A)| \leq 2^{|A|}$

Definition. *Let A be a set of regular cardinals. Then A is an interval of regular cardinals if and only if $A = [\lambda, \kappa) \cap REG$ for some λ, κ.*

There is an interesting theorem about $\mathrm{pcf}(A)$ where A is an interval of regular cardinals.

Theorem 4. *Let A be a progressive interval of regular cardinals, then* $\mathrm{pcf}(A)$ *is an interval of regular cardinals.*

This is called the no holes theorem and it is going to give us our first non-trivial bound on the size of the $\mathrm{pcf}(A)$. Moreover the proof uses the Dichotomy thereom from two weeks ago!

Recall the dichotomy theorem gives us a characterization of when a sequence of functions modulo an ultrafilter has an exact upper bound. The proof of the no holes theorem requires a technical lemma in which we build a sequence of functions and show that it has an eub using the dichotomy theorem.

Fact. *For each $\lambda \in \mathrm{pcf}(A)$, there is a set B_λ such that $J_{<\lambda^+} = J_{<\lambda} + B_\lambda$.*

One can argue that $\mathrm{cf}(\prod A/U)$ is the least λ such that $B_\lambda \in U$. Further we will prove that

Fact. *For every $\lambda \in \mathrm{pcf}(A)$, there is a sequence $\langle f^\lambda_\alpha : \alpha < \lambda\rangle$ such that for all $\alpha < \lambda$, $f^\lambda_\alpha \in \prod B_\lambda$ and the sequence is increasing and cofinal in $\prod B_\lambda / J_{<\lambda}$.*

We are abusing notation slightly when we write $\prod B_\lambda / J_{<\lambda}$, because $J_{<\lambda}$ is really an ideal on A. However, what we have written makes sense, because there is a natural ideal on B_λ generated by $J_{<\lambda}$, ie $J_{<\lambda} \restriction B_\lambda = \{X \in J_{<\lambda} : X \subseteq B_\lambda\}$.

More is true.

Fact. *If λ is least such that $U \cap J_{<\lambda^+} \neq \emptyset$, then $\langle f^\lambda_\alpha : \alpha < \lambda\rangle$, as above, is increasing and cofinal in $\prod A/U$.*

Fact. *If we fix $\langle B_\lambda : \lambda \in \mathrm{pcf}(A)\rangle$ and $\langle f^\lambda_\alpha : \alpha < \lambda\rangle$ subject to some technical considerations, then every $f \in \prod A$ is the finite pointwise supremum of functions $f^{\lambda_1}_{\alpha_1}, \ldots\ f^{\lambda_n}_{\alpha_n}$, where $\lambda_i \in \mathrm{pcf}(A)$ and $\alpha_i < \lambda_i$ for all $i \leq n$.*

Moreover the proof of these more complicated facts requires producing exact upperbounds modulo an ideal, which we saw a week ago was in the scope of the trichotomy theorem. So that's what PCF is, and we saw a little bit how the dichotomy and trichotomy fit in to the scheme of things.