Logic Seminar
Upperbounds modulo an ideal

Spencer Unger
Department of Mathematical Sciences
Carnegie Mellon University

Sept 22nd, 2009

Last time we defined a certain kind of ordering. Namely we took functions $f : X \to \text{On}$ and we ordered them using a filter, F.

$$f <_F g \Leftrightarrow \{x : f(x) < g(x)\} \in F$$

Recall filters specify large sets and so the above definition says that $f <_F g$ if and only if f beats g on a large set of coordinates. Next we had the definition of an ordering modulo an ideal, which is really the same concept. If I is an ideal then $f <_I g$ if and only if $f <_{\hat{I}} g$, where $\hat{I}$ is the dual filter, ie the set of complements of members of I.

To be explicit, we could write $f <_I g$ if and only if $\{x : f(x) \geq g(x)\} \in I$.

Last time, I also mentioned something about the relationship between certain kinds of upperbounds. Let me record the definitions here again.

Given $\langle f_i : i < \eta \rangle$ $<_I$-increasing.

1. h is an upper bound for $\vec{f}$ if and only if for all i, $f_i <_I h$.
2. h is a least upper bound if and only if h is an upper bound and for all upperbounds h', $h' \leq_I h$.
3. h is an exact upperbound if and only if h is an upper bound and for all $g <_I h$, there is $i < \eta$ such that $g <_I f_i$.

Theorem 1. *Let $X, I, \vec{f}$ be as above. Let h be an eub for $\vec{f}$ modulo I. Suppose that we have a function $H : X \to \text{On}$ such that for all $\alpha < \eta$, $g_\alpha <_I H$, then $h \leq_I H$.*

We are showing that eubs are lubs. The converse is false in general and we will give an example below.

Proof. Let $y = \{x : H(x) < h(x)\}$. We want to show that $y \in I$. Define $g : X \to \text{On}$, by $g(x) = H(x)$ for $x \in y$ and $g(x) = 0$ for $x \notin y$.

Claim. $g <_I h$.

We may assume that $f_0 \neq_I \bar{0}$. So since $f_0 <_I h$, we have $\{x : h(x) = 0\} \in I$. The claim follows, because for $x \in y$, $g(x) = H(x) < h(x)$. Because h is an eub, there is an $\alpha < \eta$, such that $g <_I f_\alpha$. By assumption we have $g_\alpha <_I H$, so we have that $g <_I H$ by transitivity. This shows that $\{x : g(x) = H(x)\} \in I$, by the definition of the ordering modulo I. This last set is just y, so $y \in I$ as required. □

Corollary 1. *If h_1, h_2 are both eubs for $\langle f_\alpha : \alpha < \eta \rangle$, then $h_1 =_I h_2$*

Proof. $h_1 \leq_I h_2 \leq_I h_1$, by the previous result. □

Now I will give an example of an lub that is not an eub. We will work modulo the ideal of nonstationary sets on ω_1. So f beats g just in case there is a club on which $f(x)$ beats $g(x)$. Let $X = \omega_1$, f_α be the constantly α function for $\alpha < \omega_1$ and $h = id_{\omega_1}$. Then h is an lub but not an eub.

Claim. *h is an lub for $\vec{f}$ mudolo $I = NS_{\omega_1}$*

Proof. First, we show that h is an upperbound. Fix $\alpha < \omega_1$ and show $f_\alpha <_I h$. Notice that $\{x : h(x) > \alpha\} = \{x : x > \alpha\}$, which is certainly a club.

Next we show that it is infact a least upperbound. Fix F an upperbound for $\vec{f}$. We claim that $h \leq_I F$. By hypothesis the set $\{x : F(x) > \alpha\}$ contains a club. Call this club C_α. Then take the diagonal intersection of the C_α.

For those of you who don't know, $\triangle_{\alpha<\omega_1} C_\alpha = \{x : x \in \bigcap_{\alpha<x} C_\alpha\}$. It is a standard fact that the diagonal intersection is a club. Let $x \in \triangle C_\alpha$, then $x \in \bigcap_{\alpha<x} C_\alpha$. This last says that for all $\alpha < x$, $F(x) > \alpha$ by the definition of C_α. So $F(x) \geq x$. So F is greater than or equal to the identity on a club, namely the diagonal intersection. So we have that h is an lub.

Now we have to show that h is not an eub. To do this we produce a function g, such that $g <_I h$, but for all $\alpha < \omega$ it's not true that $g <_I f_\alpha$.

By a theorem of Solovay, we can partition ω_1 in to ω_1 many pairwise disjoint stationary sets, call them S_α. Define g as follows. For each $\beta < \omega_1$, there is an $\alpha < \omega_1$ such that $\beta \in S_\alpha$. If $\alpha < \beta$ then let $g(\beta) = \alpha$. Otherwise, let $g(\beta) = 0$.

Claim. $g <_I h = id_{\omega_1}$

This is easy. We have defined g to be a regressive function everywhere.

Claim. *For all $\alpha < \omega_1$, it's not true that $g <_I f_\alpha$.*

Suppose not. Then there is an $\alpha < \omega_1$ and a club C in ω_1, such that for all $x \in C$, $\alpha > g(x)$. Clearly, $C' = C \setminus (\alpha + 2)$ is still club and so $C' \cap S_{\alpha+1}$ is nonempty. Let $x \in C' \cap S_{\alpha+1}$, then $g(x) = \alpha + 1$. This is impossible, because $x \in C$ so we should have had $g(x) < \alpha$. □

Keeping this example in mind we can state and prove Shelah's Trichotomy theorem. I'm guessing that I will not get to the details today, but I want to point out some of the key technical differences between this theorem and the dichotomy theorem from last time.

Theorem 2 (Trichotomy)**.** *Let I be an ideal on X and $\langle f_i : i < \lambda \rangle$ be $<_I$-increasing in ${}^X\mathrm{On}$ with $\lambda = \mathrm{cf}(\lambda) > |X|^+$. Then atleast one of the following holds:*

1. *There is an eub, h for $\vec{f}$ and for all $x \in X$, $\mathrm{cf}(h(x)) > |X|$.*
2. *There is a sequence $\langle S_x : x \in X \rangle$ with $|S_x| \leq |X|$ and an ultrafilter U with $U \cap I = \emptyset$, such that $\vec{S}$ is cofinally interleaved with $\vec{f}$ modulo U.*
3. *There is a function $g : X \to \mathrm{On}$, such that $\langle \{x : g(x) < f_i(x)\} : i < \lambda \rangle$ is not eventually constant modulo I.*

We will often call case (1) the good case, where as cases (2) and (3) are respectively called the bad and the ugly cases.

Proof. We assume that both (2) and (3) fail and produce an lub which we then argue is an eub. As in the dichotomy theorem we construct a decreasing sequence of upperbounds and argue that the construction terminates, this time in an lub.

Let $h_0(x) = \sup(f_i(x) + 1)$. If h_i is an lub then stop. If h_i is not an lub, then there is an upper bound g such that $\{x : g(x) < h_i(x)\} \in I^+$. The reason that we are not decreasing to an eub is that we are no longer working in a linear order as in the case of the dichotomy theorem. In particular, the statement h_i is not an eub modulo I is not sufficient to produce a suitable h_{i+1} for our construction and this is precisely, because we are working modulo an ideal.

Let $h_{i+1} = \min\{g, h_i\}$. So $h_{i+1} \leq h_i$ and $\{x : h_{i+1}(x) < h_i(x)\} \in I^+$. Moreover, h_{i+1} is an upper bound for the fs.

As in the dichotomy theorem the interesting part is at the limit step. Let $i < |X|^+$ be a limit ordinal. Define $S_{x,i} = \{h_j(x) : j < i\}$ and $H^i_k = \min(S_{x,i} \setminus (f_k(x) + 1))$ for each $k < \lambda$.

It is easy to see that

$$j_1 < j_2 \Rightarrow f_{j_1} <_I f_{j_2} \Rightarrow H^i_{j_1} \leq_I H^i_{j_2}$$

Claim. *If $j_1 < j_2$, then $H^i_{j_1} =_I H^i_{j_2}$ if and only if $f_{j_2} <_I H^i_{j_1}$.*

For I-almost every x, $f_{j_1}(x) < f_{j+2}(x)$. For each such x, $H^i_{j_1}(x) \leq H^i_{j_2}(x)$. So then it is clear that for each such x, $H^i_{j_1}(x) =_I H^i_{j_2}(x)$ if and only if $f_{j_2}(x) <_I H^i_{j_1}(x)$.

Key Point. *$\langle H^i_j : j < \lambda \rangle$ is eventually constant modulo I.*

This is the same kind of claim that we made in the dichotomy theorem. However, working modulo an ideal this is going to be more difficult to prove. In particular, we see why there is a third case in the theorem.

Suppose not. Let $j_1 < j_2$ and define $B_{j_1,j_2} = \{x : f_{j_2}(x) > H^i_{j_1}(x)\}$. By our assumption, for all j_1, $B_{j_1,j_2} \in I^+$ for all large enough $j_2 < \lambda$. Since condition (3) of the theorem fails, there is a set $C_{j_1} \in I^+$ such that $C_{j_1} \equiv_I B_{j_1,j_2}$ for all large $j_2 < \lambda$.

Notice that for $j < j'$, we have $H^i_j \leq_I H^i_{j'}$ and thus $C_{j'} \subseteq C_j$. It is easy to show that $\{C : \exists j < \lambda C_j \subseteq_I C\}$ is a filter. We can extend it to an ultrafilter U such that $U \cap I = \emptyset$ and for all $j < \lambda$, $C_j \in U$.

We check that the sequence of $S_{x,i}$ defined above is cofinally interleaved with $\vec{f}$ modulo U as witnessed by the H^i_ks. Let $k < \lambda$, then $f_k < H^i_k$. Then we can find k' large enough so that $C_k \equiv_I B_{k,k'}$. Then $B_{k,k'} \in U$ and so $H^i_k <_U f_{k'}$. This contradicts that we are not in case (2) of the theorem.

So $\langle H^i_k : k < \lambda \rangle$ stabilizes modulo I. Choose $h_i \equiv_I H^i_k$ for all large $k < \lambda$. Now we have finished the construction and we want to show that it halts at a successor step before stage $|X|^+$.

Suppose not. Proceed as in the proof of the dichotomy theorem.

For all x and all $k < \lambda$, for all large i, $H^i_k(x)$ is constant. Notice that for a fixed x and k, $H^i_k(x)$ is decreasing as i increases. Then as $|X|^+$ is regular and bigger than $|X|$, we have for all large i, H^i_k is constant, say for $i \geq \alpha_j$. As $\lambda > |X|^+$ there is an unbounded $A \subseteq \lambda$ and α such that for all $k \in A$, $\alpha = \alpha_k$. Choose $k \in A$ so large that $h_\alpha =_I H^\alpha_k$ and $h_{\alpha+\omega} =_I H^{\alpha+\omega}_k$. Then by the above argument we have $H^\alpha_k = H^{\alpha+\omega}_k$, which gives $h_\alpha =_I h_{\alpha+\omega}$, contradicting the fact that the sequence of h was decreasing.

Let h be a lub for $\vec{f}$.

Claim. *h is an eub.*

If not then there is a function $g <_I h$ such that for no $j < \lambda$ is $g <_I f_j$. Consider $B_j = \{x : g(x) < f_j(x)\}$. Note that $B_j \in I^+$ for j large enough. By our assumption there is no j such that B_j is equivalent to X modulo I. By the fact that case (3) fails, there is a C such that for all large j $B_j \equiv_I C$. For $x \in C$ define $k(x) = h(x)$ and for $x \notin C$ define $k(x) = g(x)$.

Then k is an upper bound, since the complement of C must be positive. If the complement of C were in I, then we would have a j such that B_j was equivalent to X modulo I. Moreover, $\neg(h \leq_I k)$ since $C \in I^+$. This contradicts that h is an lub.

Lastly we claim that $\{x : \mathrm{cf}(h(x) > |X|^+\} \in I^+$. With the claim in hand we can change h on an I-small set to get large cofinalities everywhere.

Suppose that the $B = \{x : \mathrm{cf}(h(x)) \leq |X|\} \notin I$. Then choose U with $B \in U$ and $U \cap I = \emptyset$ along with for $x \in B$, $S_x \subseteq h(x)$ cofinal and $|S_x| \leq |X|$. Then $\vec{S}$ is confinally interleaved with $\vec{f}$ modulo U, a contradiction. This concludes the proof of the trichotomy theorem. □