Logic Seminar
Eubs modulo ultrafilters

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PCF Theory stands for Possible Cofinalities theory. It was developed by Shelah to study singular cardinals and their successors. Today I am going to talk about a construction and a theorem that are essential in the study of PCF. They are the idea of ordering functions modulo an ultrafilter and Shelah's Dichotomy theorem. Shelah's Dichotomy theorem gives us a strong characterization of when there are so called exact upperbounds modulo an ultrafilter.

Let's begin with the definitions. We are going to do a little bit of an atypical introduction to PCF theoretic objects. However, I will try to spell out exactly how what we are doing today is closely related to the rest of PCF.

Definition. *A set F is a filter on a set X if and only if $F \subseteq \mathcal{P}(X)$ such that,*

1. $X \in F$
2. $A, B \in F \Rightarrow A \cap B \in F$
3. $A \subseteq B \wedge A \in F \Rightarrow B \in F$

Dual to filter we have the notion of an ideal.

Definition. *I is an ideal on X if $I \subseteq \mathcal{P}(X)$ and*

1. $\emptyset \in I$
2. $A, B \in I \Rightarrow A \cup B \in I$
3. *If $A \subseteq B \subseteq X$ and $B \in I$, then $A \in I$.*

If F is filter then the $\hat{F} = \{X \setminus A : A \in F\}$ is an ideal. Similarly for ideals.

This will be used more next week, but I would like to introduce the ideas of ordering functions in general way. It is a general fact that maximal filters exist; they will important today.

Definition. *U is an ultrafilter on X if and only if U is a filter on X and for every $A \subseteq X$, either $A \in U$ or $X \setminus A \in U$.*

Using the idea of ideal/filter, we can order certain kinds of functions. The idea that we use is kind of a special case of a model theoretic construction called a reduced product.

Let ${}^X\mathrm{On} = \{f | f : X \to \mathrm{On}\}$. Let F be a filter on X. For $f \in {}^X\mathrm{On}$, define $[f]_F = \{g \in {}^X\mathrm{On} : \{x \in X : g(x) = f(x)\} \in F\}$. Then define ${}^X\mathrm{On}/F$ to be the set $\{[f]_F : f \in {}^X\mathrm{On}\}$ together with the ordering $[f]_F <_F [g]_F$ if and only if $\{x : f(x) < g(x)\} \in F$. One can check that this is welldefined.

In practice we are going to drop the equivalence class brackets and just work with each function and write $f =_F g$ if and only if $\{x : f(x) = g(x)\} \in F$.

Though the ordinals are well ordered, if we are working modulo some random filter then the above construction need not be a linear order. By Los' Theorem, if we are working modulo an ultrafilter, then we get a linear order.

If $I = \hat{F}$ for some filter F, then we define ${}^X\mathrm{On}/I = {}^X\mathrm{On}/F$. Equivalently, we say $f <_I g$ iff $\{x : f(x) \geq g(x)\} \in I$. Think of ideals as specifying small sets and filters as specifying large sets.

For the sake of keeping some generality, I will define upperbounds, least upperbounds and exact upper bounds in the context of an ideal. Keep in mind that all of this will work if we order modulo a filter or even an ultrafilter.

Fix a limit ordinal η, we say that $\langle f_\alpha : \alpha < \eta \rangle$ is $<_I$-increasing if and only if for all $\alpha < \beta < \eta$, $f_\alpha <_I f_\beta$.

Definition. 1. *$h : x \to \mathrm{On}$ is an upperbound for $\vec{f}$ if and only if for all $\alpha < \eta$, $f_\alpha <_I h$*

2. *$h : X \to \mathrm{On}$ is an exact upper bound for $\vec{f}$ as above if and only if*

   (a) h is an eub for $\vec{f}$

   (b) $\forall g : X \to \mathrm{On}$, ($g <_I h \Rightarrow \exists \alpha < \eta$, $g <_I g_\alpha$

   We abbreviate exact upper bound as eub.

3. *$h : X \to \mathrm{On}$ is a least upperbound if and only if*

   (a) h is an upperbound

   (b) For all $g : X \to \mathrm{On}$ if g is an upperbound then $h \leq_I g$

One can show that working modulo an ultrafilter the notions of eub and lub are the same. It is easy since we are working in a linear order. Next week, I will show that working modulo an ideal, an eub is an lub, but that the converse is false.

We can now formulate a very natural question. Given a sequence $\langle f_\alpha : \alpha < \eta \rangle$ that is increasing modulo U an ultrafilter, when does this sequence have an eub modulo U?

To investigate this we need the following definition.

Definition. *Let $\langle S_x : x \in X \rangle = \vec{S}$, $S_x \subseteq \mathrm{On}$ for ech $x \in X$. $\vec{S}$ is cofinally interleaved with $\langle f_\alpha : \alpha < \eta \rangle$ if and only if for all $i < \eta$, there is $g \in \prod_{x \in X} S_x$ and j with $i < j < \eta$, such that $f_i <_U g <_U f_j$.*

With this definition in mind we can state and prove Shelah's Dichotomy theorem. Shelah's theorem gives us a precise characterization of when there is an eub modulo an ultrafilter.

Theorem 1 (Shelah's Dichotomy)**.** *Let U be an ultrafilter on X. Let $\langle f_i : i < \eta \rangle$ be increasing modulo U, with each $f_i : X \to \mathrm{On}$ where $\eta > |X|^+$ is a regular cardinal. Then either:*

1. *There is a sequence $\langle S_x : x \in X \rangle$ with $|S_x| \leq |X|$ and $\vec{S}$ is cofinally interleaved with $\vec{f}$.*
2. *There is an eub $h : X \to \mathrm{On}$ for $\vec{f}$ and $\{\ x \in X \colon \mathrm{cf}(h(x)) > |X| \} \in U$.*

Remark. A trivial modification gives an eub with large cofinalities everywhere.

Proof. Suppose that part 1 of the dichotomy fails. We will work to produce an eub by taking an upperbound and decreasing it to get what we want. We'll define a sequence $\langle h_i \rangle$ such that $i < k$ implies $h_k <_U h_i$ and for each i, $f_k <_U h_i$ for all $k < \eta$.

Define $h_0(x) = sup_{k<\eta}(f_k(x) + 1)$.

For the succesor step let h_i be given, if h_i is an eub then stop. Otherwise, there is an $h_{i+1} <_U h_i$ such that $f_k <_U h_{i+1}$ for all $k < \eta$.

Suppose that i is limit. Define $S_{x,i} = \{h_j : j < i\}$. Since i is fixed this defines a candidate for cofinal interleaving. For each $k < \eta$, define $H_{i,k} = \min(S_{x,i} \setminus (f_k(x)+1))$. These are functions in the product of the $S_{x,i}$ that might witness the cofinal interleaving(the function that we called g in the definition).

Claim. *For every $k < \eta$ and every $j < i$, $H_{i,k} <_U h_j$.*

Fix k, j as above. Then $f_k <_U h_{j+1} <_U h_j$, by the inductive hypothesis and the construction so far. So for U-almost every x, $f_k(x) < h_{j+1}(x) < h_j(x)$. So by the definition of $H_{i,k}$, we have for U almost every x, $f_k(x) < H_{i,k}(x) \leq h_{j+1} < h_j(x)$, as required.

Claim. $k_1 < k_2 < \eta \Rightarrow H_{i,k_1} \leq_U H_{i,k_2}$

This is easy using the definition of $H_{i,k}$.

Claim. *$\langle H_{i,k} : k < \eta \rangle$ is eventually constant.*

Here is where we are going to use cofinal interleaving. So far we have been working with $i < |X|^+$, so $|S_{x,i}| \leq |X|$. So we have $\langle S_{x,i} : x \in X \rangle$ is not cofinally interleaved with $\vec{f}$.

Suppose for a contradiction that for every $k_1 < \eta$, there is a $k_2 < \eta$ with $k_1 < k_2$ such that $H_{i,k_1} \neq_U H_{i,k_2}$. We will show this gives cofinal interleaving. Let $k_1 < \eta$ and pick $k_2 < \eta$ by the above. Then we have $H_{i,k_1} <_U H_{i,k_2}$, because the $H_{i,k}$ are weakly increasing modulo U. So for U-almost every x, $H_{i,k_1}(x) = \min(S_{x,i} \setminus (f_{k_1}(x) + 1)) < H_{i,k_2}(x) = \min(S_{x,i} \setminus (f_{k_2}(x) + 1))$. This shows that for U almost every x the f sequence increased between k_1 and k_2.

In particular we have for U-almost every x, we have $f_{k_1}(x) < H_{i,k_1}(x) < f_{k_2}(x)$. So we got that $\vec{S_{x,i}}$ is confinally interleaved, a contradiction.

So choose k_i such that for all $k \geq k_i$, $H_{i,k} =_U H_{i,k_i}$ and let $h_i = H_{i,k_i}$. Since for all $k < \eta$ and $j < i$ we have $f_k <_U H_{i,k} <_U h_j$, we get $f_k <_U h_i <_U h_j$. This finishes the limit stage of the construction. We would like to show that the construction terminates at a successor step, $i < |X|^+$.

Suppose for a contradiction that the construction is defined for all $i < |X|^+$. Let $T_x = \{h_j(x) : j < |X|^+\}$. Define $H_{\infty,k} = \min(T_x \setminus (f_k(x)+1))$. So for all j and all k, $f_k <_U H_{\infty,k} <_U h_j$. We also have that

$$T_x = \bigcup_{\substack{i<|X|^+ \\ i limit}} S_{x,i}$$

Fix $k < \eta$. Since the $S_{x,i}$ are increasing, for each x we can find $i_x < |X|^+$ such that for all $i \geq i_x$, $\min(T_x \setminus (f_k(x)+1)) = \min(S_{x,i} \setminus (f_k(x)+1))$. Let $\alpha_k = \sup_{x\in X} i_x$. Then $\alpha_k < |X|^+$ and for all $i \geq \alpha_k$, we get $H_{\infty,k} = H_{i,k}$.

As $\mathrm{cf}(\eta) = \eta > |X|^+$, there is an $A \subseteq \eta$ unbounded and $\alpha < |X|^+$ such that for all $k \in A$, $\alpha = \alpha_k$.

Choose i_0, i_1 limit ordinals such that $\alpha < i_0 < i_1 < |X|^+$. Consider h_{i_0} and h_{i_1}. Choose $k \in A$ large enough so that $h_{i_0} =_U H_{i_0,k}$ and $h_{i_1} =_U H_{i_1,k}$. But $H_{i_0,k} = H_{\infty,k} = H_{i_1,k}$. So we have $h_{i_0} =_U h_{i_1}$, contradicting the fact that the h_i-sequence is decreasing.

It remains to show that there is a measure one set of points where the cofinality of our eub is large. Suppose not, ie the set $\{x : \mathrm{cf}(h(x)) \leq |X|\} \in U$. For each x in the above set, choose $S_x \subseteq h(x)$ cofinal with $|S_x| \leq |X|$. For other x, choose $S_x = \{\emptyset\}$. We check that $\langle S_x : x \in X\rangle$ is cofinally interleaved with $\langle f_i : i < \eta\rangle$. Fix $i < \eta$, then $f_i <_U h$. Let $A = \{x : f_i(x) < h(x)\}$. $A \in U$. Define $g(x) = \min(S_x \setminus (f_i(x)+1))$ for $x \in A$ such that $\mathrm{cf}(h(x)) \leq \kappa$. We let $g(x) = 0$ otherwise. Notice that g is non-zero on a measure one set. So $g <_U h$ and therefore we can find $j < \eta$ such that $g <_U f_j$, as h is an eub. So $\vec{S}$ is cofinally interleaved with $\vec{f}$, a contradiction. □

For simplicity think of $X = \omega$. Consider a nonprincipal ultrafilter on ω. Given a sequence of functions, dichotomy says that either this sequence is very nicely ordered in the sense that it has an eub or it is not nicely ordered. Where not nicely ordered means something like, it is atleast as complicated as functions from ω to ω ordered by an ultrafilter extending the cofinite filter. This is a good way to think about the dichotomy theorem. Either our scale has the nice property of having an eub or it is quite complex. This complexity is captured by the property of cofinal interleaving.

This theorem has many nice consequences. One of the closest, for those who know a little more pcf theory is called the No Holes theorem. That is if A is an interval of regular cardinals then $\mathrm{pcf}(A)$ is an interval of regular cardinals.