

Talk 3 of 4 on an argument from Mitchell

Spencer Unger

February 20, 2009

1 Why not use the Levy Collapse?

We are now ready to start motivating Mitchell's proof. Let's let κ be measurable and show that we can get a generic elementary embedding with critical point $(\kappa = \omega_2)^{V[G]}$, but that this alone is not enough! The first thing that comes to mind in trying to arrange the tree property at ω_2 is the Levy collapse. Why not just collapse all of the ordinals less than κ on to ω_1 , then extend the elementary embedding to the generic extension and use the same argument as last time for κ measurable has TP? Well this proof fails for a lot of reasons, but I want to show that we can get an appropriate elementary embedding and even a branch, but this can't help us.

Proposition 1.1. *Let G be V -generic for $\text{Coll}(\omega_1, < \kappa)$. Then $V[G] \models \text{CH}$. In fact we can get a diamond sequence.*

I think that this is a modification of an exercise in our forcing class. So already from last time we know that living in $V[G]$ there is a special ω_2 -A-tree. So something must go wrong in the elementary embedding proof, but it is instructive to see exactly what goes wrong.

Let $j : V \rightarrow N$ witness κ measurable. Then by elementarity $j(\text{Coll}(\omega_1, < \kappa)) = \text{Coll}(\omega_1, < j(\kappa))_N$, which is equal to $\text{Coll}(\omega_1, < j(\kappa))_V$ by the closure of N under κ -sequences. In fact, given G $\text{Coll}(\omega_1, < \kappa)$ -generic over V , we can find $H \supseteq G$ which is $\text{Coll}(\omega_1, < j(\kappa))$ -generic over V . This is because the Levy collapse factors nicely. We have the following isomorphism, $\text{Coll}(\omega_1, < j(\kappa)) \cong \text{Coll}(\omega_1, < \kappa) \times \text{Coll}(\omega_1, [\kappa, j(\kappa)))$.

Now as the members of $\text{Coll}(\omega_1, < \kappa)$ are small and $\text{crit}(j) = \kappa$, for $p \in \text{Coll}(\omega_1, < \kappa)$, $j(p) = p$. Recall in Silver's theorem, the thing that saved us was the fact that $j''G \subseteq H$. So we have $j''G \subseteq H$ for rather trivial reasons, $G \subseteq H$ and $j \upharpoonright G$ is the identity.

So there is $j : V[G] \rightarrow N[H]$ with critical point ω_2 and all of the same things hold as before. So $N[H] \models$ "there is a branch through T." Essential in the proof of measurable κ has the tree property was that $N \subseteq V$, but note that this is no longer the case. $N[H] \not\subseteq V[G]!$ In particular, in $N[H]$ we have added a surjection from ω_1 onto κ . So κ is just some random ordinal between ω_1 and ω_2 in $N[H]$ and it is clear that this is what added the branch. The extended collapse added a branch that was not in $V[G]$.

Furthermore, we can't save the situation by adding reals after the fact with cohen forcing, as ccc forcing couldn't destroy the specialness of our ω_2 -A-tree.

So the moral of the story is that when we finally get to the end of mitchell's argument, we want to show that the extra bit forcing in j of the mitchell poset did NOT add a branch through our tree. And furthermore we need to add reals and collapse cardinals simulatenously.

2 The Structure of the Argument

Before defining Mitchell's poset, I want to say something general about how the argument is going to go. For the moment we are going to black box the forcing, which we call \mathbb{M} . We are going to end up with the same arrangement as above, because the mitchell is susceptible to a similar sort of factor analysis as the Levy collapse.

Let G be \mathbb{M} -generic over V . Then we know by Silver's theorem that we need to arrange that the pointwise image of G is contained in H , which is $j(\mathbb{M})$ -generic over V . Now this is going to work out for the following two reasons. First, $j(\mathbb{M}) \cong \mathbb{M} * j(\mathbb{M}) \upharpoonright \kappa$. We are going to get this factor analysis from a certain projection. Second, the conditions of \mathbb{M} are small relative to the critical point of j . In fact, we have for all $(p, f) \in \mathbb{M}$, $j(p, f) = (p, f)$.

What this means is that as before we can choose. A generic for $j(\mathbb{M})$ that extends G and then for trivial reasons, we have $j''G \subseteq H$. So thus we get our lifted embedding.

From our example of the Levy Collapse, we also know that we should be very interested in the forcing that takes us from our original forcing up to j of the forcing. In particular, we are going to be interested in this poset that I have called $j(\mathbb{M}) \upharpoonright \kappa$. We want to know that this forcing cannot add branches through our ω_2 -tree.

Why should we be interested that this forcing doesn't add branches? As usual we find that in the image side, $N[H]$, there is a branch through T . By our factor analysis and a general fact about projections, we can think of $N[H]$ as $N[G][H]$. So we then show that forcing with $j(\mathbb{M}) \upharpoonright \kappa$ couldn't have added branches through our tree. This shows that the branch in $N[G][H]$ is really in $N[G]$. Lastly, we have $N[G] \subseteq V[G]$, so our branch is in $V[G]$ as required.

3 Mitchell's Result

Recall Mitchell's result.

Theorem 3.1 (Mitchell). *If there exists a weakly compact cardinal κ , then there is a generic extension $V^{\mathbb{M}}$ with the following properties:*

1. $V^{\mathbb{M}} \models 2^\omega = \kappa = \omega_2$
2. ω_1 is preserved in $V^{\mathbb{M}}$.
3. ω_2 has the tree property in $V^{\mathbb{M}}$.

4 Mitchell's Poset

Definition 4.1. Let $\mathbb{P} = \text{Add}(\omega, \kappa)$ and for all $\alpha < \kappa$ let $\mathbb{P} \upharpoonright \alpha$ be $\text{Add}(\omega, \alpha)$. As I said before we will call Mitchell's poset, \mathbb{M} . \mathbb{M} is the collection of pairs, (p, f) , such that p is a finite partial function from κ to 2 and f is a function with domain a countable subset of κ and for all $\alpha \in \text{dom}(f)$, $f(\alpha)$ is a $\mathbb{P} \upharpoonright \alpha$ -name for a condition in $\text{Add}(\omega_1, 1)_{V^{\mathbb{P} \upharpoonright \alpha}}$. For the ordering, let $(p, f) \leq (p', f')$ iff $p' \subseteq p$, $\text{dom}(f') \subseteq \text{dom}(f)$ and for all $\alpha \in \text{dom}(f')$, $p \upharpoonright \alpha \Vdash_{\mathbb{P} \upharpoonright \alpha}^V f(\alpha) \leq f'(\alpha)$ in $\text{Add}(\omega_1, 1)_{V^{\mathbb{P} \upharpoonright \alpha}}$.

Note that we are going to think of $\text{Add}(\omega, \kappa)$ as $\text{Fn}(\kappa, 2)$, because they are really the same thing. So the first coordinate of Mitchell is from \mathbb{P} . Our conditions are pairs and there is some sort of interaction between the first and second coordinate. This is going to be key in the proof below. The best way to start understanding this poset is to use it to prove some easy stuff. We will warm up with parts one and two of Mitchell's theorem.

5 Sketch Proof of Mitchell's Theorem

5.1 Part 1

Let G be \mathbb{M} -generic over V . We start with proving that

$$V[G] \models 2^\omega = \kappa = \omega_2$$

We will do this in two steps. First, we show that $2^\omega = \kappa$.

Let $\pi : \mathbb{M} \rightarrow \mathbb{P}$, be given by $\pi(p, f) = p$. From our examples last time it should be obvious that π is a projection. Thus we have $V^{\mathbb{P}} \subseteq V^{\mathbb{M}}$. This shows us that $V[G] \models 2^\omega \geq \kappa$.

To see that $2^\omega = \kappa$ we use the following fact,

Lemma 5.1. \mathbb{M} is κ -cc.

The proof of this is by a usual delta system argument. We do a bunch of freezing out by fodor's lemma and create a delta system of size κ .

Using the lemma we can count nice names for subsets of ω , to find that $V[G] \models 2^\omega = \kappa$.

Next, we need to show that

$$V[G] \models \kappa = \omega_2$$

To do this we define the following projections. For each $\alpha < \kappa$, define $\pi_\alpha : \mathbb{M} \rightarrow \mathbb{P} \upharpoonright \alpha * \text{Add}(\omega_1, 1)_{V^{\mathbb{P} \upharpoonright \alpha}}$, by $\pi(p, f) = (p \upharpoonright \alpha, f(\alpha))$.

Lemma 5.2. π_α is a projection for all $\alpha < \kappa$.

Let J_α be the upwards closure of π_α " G . From the first fact about projections, we have for each $\alpha < \kappa$, $V[J_\alpha] \subseteq V[G]$ Why does this show that cardinals are collapsed? To see this we need another lemma.

Lemma 5.3. *Forcing with $Add(\omega_1, 1)$ forces CH. In particular, if $2^\omega = \eta$ in the ground model then in the extension there is a surjection from ω_1 onto η .*

By the lemma, we have for all $\alpha < \kappa$, $V[J_\alpha] \models$ "there is a surjection from ω_1 onto α ". So $V[G] \models$ "for every $\alpha < \kappa$ there is a surjection from ω_1 onto α ". Then, as \mathbb{M} is κ -cc, we know that $V[G] \models$ " κ is a regular cardinal." So $V[G] \models \kappa = \omega_2$.

5.2 Part 2

We want to show that ω_1 is preserved in our extension. To do this we are going to develop a sort of product analysis that is going to be very important to us when we argue that ω_2 has the tree property in our extension. For this set up recall that we have defined $\mathbb{P} = Add(\omega, \kappa)$.

Definition 5.4. *Let the poset \mathbb{R} be the set of second coordinates from \mathbb{M} together with the ordering, $r_1 \leq r_2$ iff $dom(r_2) \subseteq dom(r_1)$ and for all $\alpha \in dom(r_2)$, $\Vdash_{\mathbb{P} \upharpoonright \alpha} r_1(\alpha) \leq r_2(\alpha)$ in $Add(\omega_1, 1)_{V^{\mathbb{P} \upharpoonright \alpha}}$.*

Technically speaking, \mathbb{R} is the countable support product over $\alpha < \kappa$ of term forcings for $\mathbb{P} \upharpoonright \alpha * Add(\omega_1, 1)_{V^{\mathbb{P} \upharpoonright \alpha}}$.

Let $\sigma : \mathbb{P} \times \mathbb{R} \rightarrow \mathbb{M}$, be given by $\sigma(p, r) = (p, r)$.

Lemma 5.5. *σ is a projection.*

Recall our projection from last time that was the identity? So we now have the following string on inclusions.

$$V^{\mathbb{P}} \subseteq V^{\mathbb{M}} \subseteq V^{\mathbb{P} \times \mathbb{R}}$$

This is going to be tremendously useful to us, but we need to show that we are in the situation of Easton's Lemma. To this end, recall the following obvious fact.

Proposition 5.6. *$Add(\omega_1, 1)$ is countably closed.*

Then recall from last time that we had,

Proposition 5.7. *If $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is countably closed, then $A(\mathbb{P}, \dot{\mathbb{Q}})$ is countably closed.*

By the previous two propositions, it is easy to see the following,

Lemma 5.8. *\mathbb{R} is countably closed.*

So we can apply Easton's lemma. In particular this shows us that in $V^{\mathbb{P} \times \mathbb{R}}$, ω_1 is preserved. Therefore it is also preserved in $V^{\mathbb{M}}$.

The other fact that is important in the proof of the main theorem is that $\omega(V^{\mathbb{M}}) \subseteq V^{\mathbb{P}}$, because \mathbb{R} adds no ω sequences by easton's lemma.