Talk 1 of 4 on an argument from Mitchell

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As advertised, today we are going to talk about the preliminaries required for understanding a result of Mitchell. Con(ZFC + there is a weakly compact cardinal) implies Con(ZFC + ω_2 has the tree property. I'm going to begin with some of the most basic definitions, like what a tree is, and continue on to talk about some further results on the subject of Aronszajn trees. These preliminaries are going to be important for understanding the hurdles in Mitchell's proof. We begin with the definition of a tree and some general nonsense about trees.

Definition 1.1. *$(T, <_T)$ is a tree if and only if $(T, <_T)$ is a partial order, i.e. transitive and irreflexive, and for every $x \in T$, the set $\{t \in T : t <_T x\}$ is a linear ordering.*

In general, I will probably drop the sub T on the linear ordering, because it should be clear from context that we are using the tree ordering. If it is ever unclear, let me know.

Definition 1.2. *The α-th level of T, denoted $Lev_\alpha T$, is the $\{x \in T : o.t.(\{t \in T : t <_T x\}) = \alpha\}$*

Definition 1.3. *The height of T is the $\sup\limits_{x \in T} o.t.\{t \in T : t <_T x\}$*

Now we can say when a tree of height κ has a κ-branch.

Definition 1.4. *A tree T has a κ–branch if and only if there is a $B \subseteq T$ such that B is linearly ordered by $<_T$ and $|B| = \kappa$*

We are going to be interested in when certain kinds of trees have branches. Let's make that precise.

Definition 1.5. *Let κ be a regular cardinal. A tree T is a κ–tree if and only if T has height κ and for all $\alpha < \kappa$ $|Lev_\alpha T| < \kappa$*

Now we can define the notion of an Aronszajn Tree.

Definition 1.6. *A κ–tree T is Aronszajn if and only if it has no κ–branch.*

And now we define the concept that is central to our subsequent discussion.

Definition 1.7. *Let κ be a regular cardinal. κ has the Tree Property if and only if there are no κ -A-trees.*

Remark 1.8. κ has TP if and only if every κ tree has a branch.

This is is a positive charcterization that is easier to think about then the definition and is obviously equivalent.

The immediate question is, Which cardinals have the tree property and which don't? Inorder for the tree property to fail we must construct an A–tree and for it to hold we need to show that every potential A–tree actually has a branch.

So let's begin at the beginning. What about ω?

Theorem 1.9 (König Infinity Lemma)**.** *If T is a tree of height ω with $|Lev_n T| < \omega$ for all $n < \omega$, then T has an ω branch.*

Proof. We are going to pick our branch inductively. First, note that an element can't have infinitely many incomparible immediate successors, becuase then we would have an infinite level. So starting at the root, we only have finitely many choices and we pick the next element of the branch to be an element with infinitely elements above it. There must be such an element becuase otherwise the tree would be finite and thus couldn't have height ω. Clearly we can continue to pick in this way and this gives us a branch. □

Corollary 1.10. *ω has the tree property*

What about ω_1?

Theorem 1.11 (Aronszajn)**.** *There is an ω_1-A-tree.*

Sketch. We use bounded increasing sequences of rational numbers, because this proof lends itself to generalizations, which we will see later. Key in the proof is the fact that the rationals are dense. So if $x, y \in {}^{\omega}\mathbb{Q}$ are bounded, then we let $x \leq y$ if and only if $x \subseteq y$. T is going to be downwards closed so if $y \in T$ and x is an initial segment of y, then $x \in T$. By using $\mathbb{Q}$ and the ordering as defined we are ruling out the possibility of a branch as there is no increasing sequence of ω_1 rationals. So T will be Aronszajn provided that we can give T height ω_1 and have all of T's levels be countable.

To this end we arrange the following inductive assumption:
For every $\beta < \alpha$, $x \in Lev_\beta T$, and $q \in \mathbb{Q}$ with $q > \sup x$, there exists a $y \in Lev_\alpha T$ such that $x \subseteq y$ and $\sup y \leq q$.

Note that we want to make sure that each new level is countable. So at the successor step we add one element to the top of each sequence in the previous level to get an element on the new level. This takes the sequence of order type α to a sequence of order type $\alpha + 1$. And we can do this while satisfying the inductive hypothesis, because the rationals are dense. For a limit α, we fix an $x \in Lev_\beta T$ for $\beta < \alpha$. We know that the cofinality of α is ω and we pick a cofinal omega sequence of ordinals beginning at β. Then by repeated use of the inductive assumption we grow a sequence of rationals of order type α along this sequence of ordinals. So the element on the limit level will be a union of our sequence of sequences. We have to be careful about the bound of our sequence, but again this is going to work because the rationals are dense. And this does it. □

Next, we have a generalization of the previous result.

Theorem 1.12 (Specker)**.** *If $\kappa^{<\kappa} = \kappa$ then there is a κ^+-A-tree.*

Super sketch. We use the cardinal arithmetic assumption to construct a saturated linear order and then the proof is almost word for word the same as the proof for the rationals except that our confinal sequence of ordinals is going to be longer. $\square$

Example 1.13. CH $\Rightarrow \aleph_1^{\aleph_0} = \aleph_1 \Rightarrow \aleph_1^{<\aleph_1} = \aleph_1 \Rightarrow$ there is an ω_2-A-tree.

This is going to be a very important example for us. The above A-trees, with a slight modification in the construction, are special in the following sense.

Definition 1.14. *A κ^+-A-tree is special if and only if there is a function $f : T \to \kappa$ such that if $x, y \in T$ and $x < y$, then $f(x) \neq f(y)$.*

We are going to switch topics a little, moving from constructing Aronszajn trees to arranging the tree property, ie that there are no Aronszajn trees.

Here is where large cardinals come in to the picture. Many large cardinals have the tree property. In fact, the tree property 'starts' at weakly compact cardinals. Let's make that more precise.

Definition 1.15. *An innacessible cardinal κ is Weakly Compact if and only if for every transitive set $M \models ZFC-P$ with $|M| = \kappa$, $H(\kappa) \subseteq M$ and ${}^{<\kappa}M \subseteq M$, there is a nontrivial elementary embedding $k : M \to N$ for some transitive set N with crit(j)= κ.*

Theorem 1.16. *A cardinal κ is weakly compact if and only if κ is strongly inaccessible and has the tree property.*

Recall: κ is strongly inaccessible if and only if it is a regular strong limit cardinal.

I think that most people here have seen this theorem. It could go many different ways depending on the definition of weakly compact that you choose. The relevant result for us is the following:

Theorem 1.17. *If κ is a measurable cardinal then κ has the tree property.*

Recall

Definition 1.18. *A cardinal κ is measurable if and only if there is a nontrivial elementary embedding $j : V \to N$ with crit(j)= κ and ${}^{\kappa}N \subseteq N$ for some transitive class N.*

Proof. Let j witness κ measurable. By elementarity, N thinks that $j(T)$ is a $j(\kappa)$-tree. By elementarity and because crit$(j)= \kappa$, for all $\alpha < \kappa$ $Lev_\alpha T = Lev_\alpha j(T)$. So $j(T) \restriction \kappa = T$. As N thinks $j(T)$ is a $j(\kappa)$-tree, then N thinks $Lev_\kappa j(T) \neq \emptyset$. So N can read of a branch through T, it is just the set of predecessors of some point on level κ of $j(T)$. This branch is definable in N and it really is a branch through T in V, because the ordering on T is absolute between V and N. $\square$

This more or less ends the 'easy' results about the tree property. From here on I am going to survey some of the major consistency results that continue this line of research. I will begin by stating Mitchell and a dual result to Mitchell's due to Silver.

Theorem 1.19 (Silver). *If ω_2 has the tree property then ω_2 is weakly compact in L.*

Theorem 1.20 (Mitchell). *If there exists a weakly compact cardinal κ, then there is a generic extension $V^{\mathbb{M}}$ with the following properties:*

1. $V^{\mathbb{M}} \models 2^{\omega} = \kappa = \omega_2$
2. *ω_2 has the tree property in $V^{\mathbb{M}}$.*

Together these give us that the tree property at ω_2 and a weakly compact cardinal are equiconsistent. Over the next two talks, I will present a sketch proof of Mitchell's argument where we assume the existence of a measurable cardinal. This added assumption will ease the discussion somewhat.

Let's survey some of the more difficult results in this line of research.

Theorem 1.21. *(Magidor) If ω_2 and ω_3 have the tree property then $0^{\sharp}$ exists.*

Magidor's result tells us that if we want to have the tree property at two succesive cardinals then we are going to need more than just two weakly compact cardinals.

In fact, much more is true in this line of thinking.

Theorem 1.22 (Foreman, Magidor, Schindler). *If ω_n has the tree property for all n with $2 \leq n < \omega$, then for all $X \in H(\aleph_\omega)$, $M_n^{\sharp}(X)$ exists.*

We begin with a definition.

Definition 1.23. *A cardinal κ is λ-supercompact if and only if there is an elementary embedding $j : V \to N$ with crit(j)= κ, $\lambda < j(\kappa)$ and ${}^{\lambda}N \subseteq N$. And κ is supercompact if and only if it is λ-supercompact for all λ*

Theorem 1.24 (Foreman and Cummings). *If there is an ω-sequence of supercompact cardinals then there is a forcing extension in which $\aleph_n$ has the tree property for all $2 \leq n \leq \omega$.*

Theorem 1.25 (Magidor and Shelah). *If λ is the singular limit of λ^+-supercompact cardinals then λ^+ has the tree property.*

Which is used in the proof of the following:

Theorem 1.26 (Magidor and Shelah). *From very large cardinal assumptions, it is consistent that $\aleph_{\omega+1}$ has the tree property.*

Interestingly, in the Magidor, Shelah construction $2^{\aleph_\omega} = \aleph_{\omega+1}$. And by Specker, this gives a special $\aleph_{\omega+2}$-A-tree. This is a problem if we want to arrange the tree property at both $\aleph_{\omega+1}$ and $\aleph_{\omega+2}$. Itay Neeman's result is a good first step in this direction.

Theorem 1.27 (Neeman)**.** *Assuming that there are ω many supercompact cardinals, then it is consistent that there is a cardinal κ with the following properties:*

1. *κ is a strong limit cardinal of cofinality ω.*
2. *$2^\kappa = \kappa^{++}$.*
3. *κ^+ has the tree property.*

The first two conditions imply that the Singular Cardinal Hypothesis fails at κ.

That is all that I have for this first talk. Next week I will talk about the general forcing facts needed for the Mitchell construction with κ measurable and hopefully end with the motivation for and the definition of Mitchell's poset. The last talk will be devoted to a sketch of why the construction works. I of course reserve the right to change the plan as needed.