INTRODUCTION TO LARGE CARDINALS

SPENCER UNGER

Abstract. The purpose of this talk is to give the non-set theorist an introduction to large cardinals. Little set theory will be assumed and some basic set theoretic concepts will be reviewed. The talk has two main aims. The first is to present a few large cardinal assumptions as direct generalizations of properties of the natural numbers. The second is to try to gain some insight in to the question 'Why are large cardinals large?'

The aim of this talk is to give the non set theorist an introduction to large cardinals. This is going to be a little bit tricky. Large cardinals are inherently set theoretic things. The point of this talk is to show that some basic large cardinal axioms can be obtained by generalizing combinatorial properties of the natural numbers to uncountable sets. Here is a rough outline of what I'm going to say.

(1) Review the axioms of set theory.
(2) An introduction to ordinals and cardinals.
(3) Sets, classes and inaccessible cardinals.
(4) Some properties of the natural numbers.
(5) Generalizations to uncoutable sets.
(6) Why are large cardinals large?

1. Axioms of Set Theory

This is sometimes a scary topic for people. However, many of the axioms of set theory are just boneheaded statments about what should be true about sets. Here are some boneheaded axioms with boneheaded formulations.

(1) A set is determined by its elements.
(2) If A, B are sets, then $\{A, B\}$ is a set.
(3) If I can define a function whose domain is a set, then the range is also a set.
(4) Unions exist (Logicians are a little picky about how this is stated so I'll omit it.)

Date: December 6th, 2011.

(5) The collection of members of a set statisfying a property is a set.
(6) The collection of all subsets of a set is a set.

So what are we missing. From here we get less boneheaded, but we have more controversy in two cases and less intuition in another.

(1) There is an infinite set. (This one really opens up a large can of worms. Emphasis on large.)
(2) There are no infinite decreasing $\in$-chains (????)
(3) The axiom of choice.

Not too many people dispute the axiom of infinity. Regularity is a little confusing, but if you think about it this seems like a natural property to demand of the membership relation. And I'm not really going to say anything about choice. The only thing I'll say is that set theory can be really weird without choice.

So this was worth mentioning I hope, because a very basic fact about large cardinals is that they are *extra* assumptions. They provably go beyond the axioms of ZFC. Maybe more on this later. We'll end this section with a fact.

Fact 1.1. *The set of all sets is not a set.*

2. Ordinals and Cardinals

A few definitions.

Definition 2.1. *A set X is transitive if and only if for every x, y, if $x \in y \in X$, then $x \in X$.*

This is a definition that comes up all the time in set theory and practically nowhere else that I've seen.

Definition 2.2. $(W, <)$ *is a well ordering if and only if it is a linear ordering with no infinite decreasing chain.*

Set theorists like things with 'well' in front of them. We'll see why in a second.

Definition 2.3. *A set α is an ordinal if and only if α is transitive and $(\alpha, \in)$ is a well order.*

Some facts about ordinals.

Fact 2.4. *Every well ordering is isomorphic to an ordinal.*

Fact 2.5. *If α is an ordinal, then $\alpha \cup \{\alpha\}$ is an ordinal. The union of a set of ordinals is an ordinal.*

Fact 2.6. *The collection of all ordinals is not a set, but it is transitive and well ordered by* $\in$.

We can define the collection of all ordinals. We'll call it On. So $x \in On$ will be short hand for 'x is an ordinal'. So if On were a set, then it would be an ordinal. Ordinals are the spine of the set theoretic universe.

Definition 2.7. *An ordinal* κ *is a cardinal if and only if there is no surjection from an ordinal* $\beta < \kappa$ *on to* κ.

(This is probably ok, since I'm assuming choice.)

Fact 2.8. *(AC) Every set is in bijection with a unique cardinal.*

Nice properties of cardinals

Definition 2.9. *A cardinal* κ *is called* singular *if it is the union of fewer than* κ *many cardinals less than* κ

Definition 2.10. *A cardinal* κ *is regular if it is not singular*

So being a regular cardinal is a nice property. It says that the only way to approach κ from below is via κ-many steps.

Let's think about how we create larger and larger cardinals in set theory.

Fact 2.11. *For every ordinal there is a cardinal.*

Proof. This is sort of abstract, but it follows from what I've said so far. Suppose that κ is a cardinal. Then the cardinality of $\mathcal{P}(\kappa)$ is greater than κ by a well known theorem of Cantor. Let κ^+ be the least cardinal greater than κ. We know there is one and the ordinals are wellfounded so there is a least one.

Suppose that κ_α for $\alpha < \beta$ is an increasing sequence of cardinals. Then $\mu = \bigcup_{\alpha<\beta} \kappa_\alpha$ exists using replacement and the union axiom. It is not hard to show that μ is a cardinal using the definition. □

Fact 2.12. *The collection of all cardinals* Card *is not a set.*

3. Sets,Classes and Inaccessible Cardinals

So we've seen some examples of sets and classes. However, it turns out that what is a set and what is a class depends on the model of Set theory you are working in.

We proved in the last section that there are an essentially limitless number of ordinals. The question that is going to lead us to inaccessibility is 'What happens in the limit of this process of constructing ordinals?' We can just go on forever finding larger and larger cardinals/ordinals until we construct things that are not sets.

This is a little unsatisfying. I can talk about the class of ordinals or the class of cardinals, but I cannot use them as I would sets. So here is an additional albeit informal hypothesis that I might make.

Informal Hypothesis 3.1. On *is a set.*

This is not really a great leap. We've talked about On (we defined it) and we said that if it were a set it would be an ordinal.

Let's explore what properties the set On would have. Notice that $\mathrm{On} = \bigcup_{\alpha \in \mathrm{On}} \alpha = \bigcup_{\alpha \in \mathrm{Card}} \alpha$. We said above that the union of a st of cardinals is a cardinal, so On is a cardinal!

Moreover it is a regular cardinal. For any increasing sequence of fewer than On many cardinals it's supremum is still less than On, since it's supremum is one of the cardinals we constructed when we first showed there was a class of cardinals.

Moreover for each cardinal $\kappa < \mathrm{On}$, $|\mathcal{P}(\kappa)| < \mathrm{On}$, since $\mathcal{P}(\kappa)$ is one of the cardinals from our original construction. This property of a cardinal is called being *strong limit*.

Formal Hypothesis 3.2. *There is an uncountable regular strong limit cardinal.*

Such cardinals are called *inaccessible*. We'll state the most important consequence of the existence of an inaccesible cardinal.

Let's define everything.

$$V_0 = \emptyset$$
$$V_{\alpha+1} = \mathcal{P}(V_\alpha)$$
$$V_\beta = \bigcup_{\alpha<\beta} V_\alpha$$
$$V = \bigcup_{\alpha \in \mathrm{On}} V_\alpha$$

V is a class and so the definition in the fourth line is a fake. We cannot take the union over all ordinals. If V were a set, then it would be a model of set theory. We've run in to this before in a slightly different way with On.

Fact 3.3. *If* κ *is an inaccessible cardinal then* $(V_\kappa, \in)$ *is a set model of* ZFC

It is not hard to show that the statement 'There is an inaccessible cardinal κ' is independent of ZFC. So ZFC plus 'there is an inaccessible cardinal' is really a stronger set of axioms than just ZFC. To end this section let me justify a previous remark. Suppose that κ is an inaccesible cardinal. Then working in $(V_\kappa, \in)$, κ is a proper class. While in the real world, we know that κ is a set. So in the presence of large cardinals the notion of class is relative to the model of set theory in which we're working.

4. Properties of ω

For this talk $\omega = \mathbb{N} = \{0, 1, 2, 3 \dots\}$. We take the elements $n \in \omega$ to be the finite ordinals. By doing this we see that ω is a cardinal! Moreover,

Fact 4.1. *ω is a strong limit cardinal.*

So ω is almost a large cardinal except that it is countable... Let's talk about some other properties of ω.

Fact 4.2. *For every 2-coloring of unordered pairs from ω, there is an infinite homogeneous set.*

This is just infinite Ramsey theorem. Let's prove it.

Proof. Suppose that $f : [\omega]^2 \to 2$. Note that 2 is an ordinal, so $2 = \{0, 1\}$. For each $n \in \omega$ there is a function $f_n : (\omega \setminus n) \to 2$ given by $f_n(m) = f(n, m)$. (We'll assume that writing $f(n, m)$ means that $n < m$.)

Define an infinite sequence of natural numbers, $\langle k_n \mid n \in \omega\rangle$ and infinite sequence of subsets of the natural numbers $\langle H_n \mid n < \omega\rangle$. Let $k_0 = 0$ and $H_0 = \omega$. Suppose we've defined k_n and H_n. Consider f_{k_n}. There is an infinite $H_{n+1} \subseteq H_n$ such that f_{k_n} is constant on H_{n+1}. Let k_{n+1} be the least element of H_{n+1} and record the color i_n.

Consider the map $n \mapsto i_n$. It is constant on an infinite $L \subseteq \omega$. It is not hard to see that $\{k_n \mid n \in L\}$ is homogeneous. □

Fact 4.3. *There is a nonprincipal ultrafilter on ω.*

Definition 4.4. *An ultrafilter $\mathcal{U}$ on X is principal if there is an $a \in X$ such that $\mathcal{U} = \{A \subseteq X \mid a \in A\}$.*

So principal ultrafilters are trivial in some sense. Nonprincipal ultrafilters are not principal. Using Zorn's lemma we can see that every filter can be extended to a maximal filter. Maximal filters are ultrafilters. So it is enough to start with a filter that will end us up with a nonprincipal ultrafilter. It is straightforward to see that $\mathcal{F} = \{A \subseteq \omega \mid A$ is cofinite $\}$ always extends to a nonprincipal ultrafilter.

5. Large Cardinal Properties

We've already discussed inaccessible cardinals. The closest that I'll get to giving a definition of large cardinals is that a cardinal is large if it is inaccessible. In this section we'll define some large cardinal properties using the facts we proved about ω.

Definition 5.1. *A cardinal κ is weakly compact if it is uncountable and for every 2-coloring of unordered pairs from κ, there is a κ sized homogeneous set.*

This is exactly the property that we had before but we've required that our cardinal be uncountable. For the next one I've been slightly dishonest.

Definition 5.2. *An ultrafilter $\mathcal{U}$ is κ-complete if it is closed under intersections of fewer than κ many sets.*

Definition 5.3. *A cardinal κ is measurable if it is uncountable and there is a nonprincipal, κ-complete ultrafilter on κ.*

So if we removed uncountability ω would be measurable, since ω-complete just means closed under finite intersections (which is true of all filters).

6. Why are large cardinals large?

So we've seen that inaccessible cardinals are large in the sense that they are the limit of some process applied to cardinals below. We'll end with showing that weakly compact are inaccessible and that measurable cardinals are weakly compact. This will justify calling in weakly compact and measurable cardinals large.

Recall that κ is inaccessible if it is an uncountable regular strong limit cardinal. For the first proof we'll need a fact which I won't prove.

Fact 6.1. *Let μ be a cardinal. There is a coloring of pairs from 2^μ with no homogeneous set of size μ^+.*

It essentially comes down to the fact that if we order 2^μ lexicographically, then there is no chain of length μ^+.

Fact 6.2. *If κ is weakly compact, then it is inaccessible.*

Proof. It is enough to show that κ is regular and strong limit. Suppose for a contradiction that κ is singular. In particular assume that $\kappa = \bigcup_{\alpha<\beta} \kappa_\alpha$ for some $\beta < \kappa$ and κ_α's for $\alpha < \beta$.

We'd like to define a coloring to get a contradiction. For $\delta < \gamma < \kappa$, define $f(\delta, \gamma) = 0$ if there is an $\alpha < \beta$ such that $\delta < \kappa_\alpha < \gamma$ and let $f(\delta, \gamma) = 1$ otherwise. We get a set $H \subseteq \kappa$ of size κ which is homogeneous for f.

Suppose that H is 1-homogeneous. Let δ be the least member of H. Let α be least such that $\delta \in \kappa_\alpha$. It follows that for all $\gamma \in H$, $\gamma \in \kappa_\alpha$ by homogeneity. This is impossible, since then we'd have κ-many distinct members of κ_α which is less than κ.

Suppose that H is 0-homogeneous. For each $\gamma \in H$, let α_γ be the least ordinal such that $\gamma \in \kappa_{\alpha_\gamma}$. By homogeneity it follows that $\gamma \mapsto \alpha_\gamma$ is a one-to-one map from a set of size κ in to β, which is less than κ. This contradicts that κ is a cardinal.

The fact that I didn't prove will allow us to show that κ is strong limit. Assuming that there is a μ such that $2^\mu = |\mathcal{P}(\mu)| \geq \kappa$. We can find a coloring of $[\kappa]^2$ with no homogeneous set of size μ^+. □

For the second fact I need an extra property and another fact that I won't prove.

Definition 6.3. *An ultrafilter $\mathcal{U}$ on κ is normal if for every sequence of sets A_α for $\alpha < \kappa$ with $A_\alpha \in \mathcal{U}$ for all $\alpha < \kappa$, the set $\{\beta \mid \beta \in \bigcap_{\alpha<\beta} A_\alpha\} \in \mathcal{U}$.*

Fact 6.4. *If there is a nonprincipal κ-complete ultrafilter on κ, then there is normal nonprincipal κ-complete ultrafilter on κ.*

We'll try to repeat the proof that we gave for ω with the use of our nice ultrafilter. Note if $\mathcal{U}$ winesses that κ is measurable, then $\kappa \setminus \alpha \in \mathcal{U}$ for all $\alpha < \kappa$.

Fact 6.5. *If κ is measurable, then κ is weakly compact*

Proof. Let $\mathcal{U}$ be a normal, nonprinicipal, κ-complete ultrafilter on κ. Let $f : [\kappa]^2 \to 2$. For each $\alpha < \kappa$, define f_α by $f_\alpha(\beta) = f(\alpha, \beta)$.

For each $\alpha < \kappa$, there is a measure one set A_α and $i_\alpha \in 2$, such that f_α is constant on A_α with value i_α. We can do this since f_α partitions the measure one set $\kappa \setminus \alpha$ into 2 pieces. One of these pieces must be measure one, since $\mathcal{U}$ is an ultrafilter.

Let $D = \{\beta \mid \beta \in \bigcap_{\alpha<\beta} A_\alpha\} \in \mathcal{U}$. D is the disjoint union of $\{\beta \in D \mid i_\beta = 0\}$ and $\{\beta \in D \mid i_\beta = 1\}$. One of these is $\mathcal{U}$-measure one. Name the set H and let $i \in 2$ be such that $i = i_\beta$ for all $\beta \in H$.

Suppose that $\alpha < \beta$ are both in H. Since $\alpha, \beta \in D$, $\beta \in A_\alpha$. It follows that $f(\alpha, \beta) = i_\alpha = i$. It is easy to show that all measure one sets have size κ using κ-completeness. So we're done. □

We could have given a proof without normality that even more closely resembles the proof of infinite Ramsey theorem, but notice we proved the stronger claim that there is a homogeneous measure one set.

That's it, thanks!