HW FOR MATH 235, SPRING 2022

Lecture 1.

(1.1) Complete the details of the proof that the projection of a small rotation of any link is a link diagram.

(1.2) Prove that the different variants of Reidemeister I and Reidemeister III moves can be obtained from one another using Reidemeister II moves.

(1.3) Prove that 3-colorability is invariant under the Reidemeister III move.

(1.4) Using Wirtinger presentation, prove 3-colorability is equivalent to knot group having a surjection to S_3.

Lecture 2.

(2.1) Prove that connected sum, as defined diagrammatically in the lecture, is a well-defined operation on oriented knots.

Lecture 3.

(3.1) Compute Alexander polynomial of the positive trefoil $T_{2,3}$ using the Seifert matrix A from a genus 1 Seifert surface. Also compute it using the skein relation.

(3.2) Compute Jones polynomial of the positive trefoil using the skein relation.

(3.3) Prove that skein relations imply uniqueness (although, not well-definedness). That is, given polynomials $f, g, h \in \mathbb{Z}[t, t^{-1}]$, prove that there is at most one polynomial-valued link invariant P satisfying $P(U) = 1$ and

$$fP(L_+) + gP(L_-) = hP(L_{\mathrm{or}})$$

for all oriented skeins $L_+, L_-, L_{\mathrm{or}}$.

Lecture 4.

(4.1) Prove that the powers of q in the Jones polynomial have the same parity as the number of link components.

(4.2) Compute the Jones polynomial of the positive and negative Hopf links using the Kauffman cube of resolutions.

(4.3) Prove Jones polynomial, as defined via the Kauffman cube of resolutions, is invariant under Reidemeister moves.

Lecture 5.

(5.1) Complete the computation of Khovanov homology of the positive trefoil.

Lecture 6.

(6.1) Check that the proofs of Reidemeister I and II invariance respect the bigradings on the Khovanov chain complex.

(6.2) Complete the proof of Reidemeister III invariance.

Lecture 7.

(7.1) Prove that the number of components in a resolution of a connected link diagram D equals the number of components in the corresponding spanning subgraph of the black graph plus the number of components in the corresponding spanning subgraph of the white graph minus 1.

(7.2) Prove that connected resolutions of a connected link diagram D correspond to spanning trees of the black graph.

Lecture 8.

(8.1) Given a pointed link diagram (D, p), give a complete proof that the chain homotopy type of $CKh(D)$ over $\mathbb{Z}[X]/X^2$ is an invariant of the underlying pointed link. (First prove invariance under Reidemeister moves away from p.)

Lecture 9.

(9.1) Prove that $\mathbb{Z}[h,t][X]/(X^2 = hX + t)$ with counit

$$1 \mapsto 0, \quad X \mapsto 1$$

and comultiplication

$$1 \mapsto 1 \otimes X + X \otimes 1 - h1 \otimes 1, \quad X \mapsto X \otimes X + t1 \otimes 1$$

is a Frobenius algebra.

(9.2) Prove that $C, D \in \mathrm{Kom}(\mathcal{C})$ are chain homotopy equivalent if and only if their images in the homotopy category $K(\mathcal{C})$ are isomorphic.

Lecture 10.

(10.1) Write down the explicit chain maps and chain homotopies for the RII invariance in the original Khovanov homology.

(10.2) Using the above exercise as a guide, in the Bar-Natan picture world, write down explicit chain maps and chain homotopies for the RII invariance.

Lecture 11.

(11.1) Complete the check that $\mathbb{Z}[h, h^{-1}][X]/(X^2 = hX)$ and $\mathbb{Z}[\frac{1}{2}][\sqrt{t}, \sqrt{t}^{-1}][X]/(X^2 = t)$ are diagonalizable Frobenius algebras.

(11.2) Prove that for any diagonalized Frobenius algebra, the Khovanov generator g_o corresponding to an orientation o of the link lies in homological grading $2\mathrm{lk}(L_{\mathrm{agree}}, L_{\mathrm{disagree}})$, where L_{agree} (respectively L_{disagree}) is the sublink where o agrees (respectively disagrees) with the given orientation of L.

Lecture 12.

(12.1) Prove that any f.g. free chain complex over $\mathbb{F}[t]$ (with $\mathrm{gr}(t) \neq 0$) is chain homotopy equivalent to a direct sum of copies of $\mathbb{F}[t]$ and copies of two step complexes $(\mathbb{F}[t] \xrightarrow{t^k} \mathbb{F}[t])$ with $k > 0$.

(12.2) Using Knot Atlas to get $Kh(5_1, \mathbb{Q})$, write down a decomposition of the its Khovanov chain complex over $\mathbb{Q}[t]$ of the above form.