RESEARCH STATEMENT

STEDMAN WILSON

Please visit my website, `http://www.math.ucla.edu/~stedmanw/research`, for more information, 3D figures, and links to submitted papers.

My research advisor is Igor Pak, and my area of specialization is discrete geometry. My research focuses on polyhedral complexes and triangulations. A natural question arises: When does a topological polyhedral complex (embedded in $\mathbb{R}^d$) admit a geometric realization (a rectilinear embedding in $\mathbb{R}^d$)? What are the constraints on the possible realizations?

A starting point for our investigations in 3-dimensions is the classical *Steinitz theorem* (see e.g. [G, P, R, Z1]). It states that all 3-connected planar graphs can be realized as graphs of 3-polytopes. As a consequence of the proof, all polytopes in $\mathbb{R}^3$ can be realized over $\mathbb{Q}$ (i.e. realized with rational vertex coordinates) by applying small perturbations of the vertices which preserve combinatorial structure (the faces of the polytope). There are several directions into which this result has been shown to have negative analogues:

(1) in $\mathbb{R}^d$, $d \geq 4$, there exist irrational convex polytopes (see [R, RZ]),

(2) there exists a 3-dim topological simplicial complex that is not geometrically realizable (see [C, HZ, K]).

In light of Steinitz's theorem and (1), it is natural to ask whether every geometrically realizable 3-dim *polyhedral complex* (a family of 3-polytopes attached face-to-face) can in fact be realized over $\mathbb{Q}$. A 3-dim polyhedral complex is a natural generalization of a Schlegel diagram of a 4-polytope, so this question occupies an intermediate position between Steinitz's theorem and (1). We have answered this question in the negative:

Theorem 0.1 *In $\mathbb{R}^3$, there exists an irrational 3-dim geometric polyhedral complex consisting of 1278 convex polyhedra.*

The proof of Theorem 0.1 follows the same general approach as (1), going back to Perles's first original construction of an irrational polytope in $\mathbb{R}^8$ [G]. We start with an irrational point and line configuration in the plane, and then use polyhedral gadgets to constrain the realization space emulating the configuration. In the end, we obtain an explicit construction consisting of 1277 triangular prisms and one pentagonal pyramid.

In order to prove (2), the results of [HZ, K] (see also [AB, C, Wi]) rely on topological simplicial complexes whose 1-skeletons contain a nontrivial knot with 5 or fewer edges (this is an obstruction to a rectilinear realization). By a much simplified variation on a construction from the proof of Theorem 0.1, we have shown that if one replaces "simplicial" with "polyhedral", one obtains very small examples of not geometrically realizable complexes, much smaller than those in [HZ, Wi]:

Theorem 0.2 *There exists a topological 3-dim polyhedral complex with 8 vertices and 3 polyhedra, that is not geometrically realizable.*

Heuristically, both (2) and Theorem 0.2 say that one cannot hope to extend the Fáry and Tutte theorems to $\mathbb{R}^3$ in full generality. Despite this, we have obtained a positive result which complements Theorems 0.1 and 0.2, and applies to a natural class of polyhedral complexes. Our full result is somewhat involved, so we state it here only for simplicial complexes.

We restrict ourselves to simplicial complexes which are homeomorphic to a ball, and which are *vertex decomposable* (see e.g. [BP, Wo]). This is a topological property that implies *shellability*. Recursively, a d-dimensional simplicial ball X is vertex decomposable if it has a boundary vertex $v \in \partial X$, such that the the deletion $X \smallsetminus v$ is also a vertex decomposable d-ball. We say that X is *strongly* vertex decomposable if in addition, the vertex v is adjacent to exactly d boundary edges.

Theorem 0.3 *Let $d \leq 3$ and let X be a topological d-dim simplicial complex that is homeomorphic to a ball and strongly vertex decomposable. Then there is a geometric simplicial complex Y such that Y is a realization of X over $\mathbb{Q}$.*

We have also found a new proof of the Steinitz theorem in the case of plane triangulations, and an extension of the result to all 2-connected plane triangulations. As a consequence, we obtain a new general bound on the grid size of the simplicial polytope realizing a given triangulation, which is subexponential in a number of special cases.

One such special case is that of a *grid triangulation*, which is a triangulation of $[a \times b]$ with all grid points as the set of vertices. These triangulations have a curious structure, and have been studied and enumerated in a number of papers (see [A, Du, KZ, W] and references therein). In the case of grid triangulations we obtain the following result:

Theorem 0.4 *Let G be a grid triangulation of a $[p \times q]$ grid such that every triangle fits in an $[\ell \times \ell]$ subgrid. Then G can be realized as the graph of a convex polyhedral surface with vertices lying in an integer grid of size $O(pq) \times O((pq)^2) \times \exp O(\ell(p+q))$.*

For example, with $p = q = \sqrt{n}$, and $\ell = O(1)$, we have a subexponential grid size for the resulting polyhedral surface: $O(n) \times O(n^2) \times \exp O(\sqrt{n})$.

I am currently working on a generalization of the Steinitz theorem that provides a correspondence between the graphs of 3-dimensional convex caps and a certain class of 2-connected plane graphs. The methods used to prove Theorem 0.4 already achieve this in the special case of simplicial polyhedra and plane triangulations. It is also my dream (perhaps an overly optimistic one) that with some tweaking, these methods could be used to achieve subexponential upper bounds on the integer grid size for *all* simplicial polytopes.

I am also fleshing out the details of how and when *any* point and line configuration can be encoded into a 3-dim polyhedral complex, using the techniques of the construction of Theorem 0.1. The result is a type of "universality theorem" for 3-dim polyhedral complexes. Another interesting direction for further research, related to Theorem 0.1, is the possibility of using similar polyhedral gadgets to create polyhedral complexes in 3 and 4 dimensions whose realization spaces have very small dimension, compared to what one would expect in the "typical" case.

References

[A] E. E. Anclin, An upper bound for the number of planar lattice triangulations, *J. Combin. Theory Ser. A* **103** (2003), no. 2, 383–386.

[AB] K. Adiprasito and B. Benedetti, Metric Geometry and Collapsibility, `arXiv:1107.5789`.

[BP] L. Billera and J. Provan, Decompositions of simplicial complexes related to diameters of convex polyhedra, *Math. Op. Res.* **5** (1980), 576–594.

[BR] I. Bárány and G. Rote, Strictly convex drawings of planar graphs, *Documenta Math.* **11** (2006), 369–391.

[BS] K. Buchin and A. Schulz, On the number of spanning trees a planar graph can have, in *Proc. ESA 2010*, 576–594.

[C] S. S. Cairns, Triangulated manifolds which are not Brouwer manifolds, *Ann. Math.* **41** (1940), 792–795.

[CFGN] G. Chapuy, É. Fusy, O. Giménez and M. Noy, On the diameter of random planar graphs, in *Proc. DMTCS 2010*, 65–78.

[DRS] J. A. De Loera, J. Rambau and F. Santos, *Triangulations: Structures and Algorithms*, Springer, 2008.

[DS] E. D. Demaine and A. Schulz, Embedding stacked polytopes on a polynomial-size grid, in *Proc. SODA 2011*, 1177–1187.

[Du] J. Dunkel, How many triangulations exist for a 6×6 grid?, preprint (2003).

[F] I. Fary, On straight line representations of planar graphs, *Acta Univ. Szeged. Sect. Sci. Math.* **11** (1948), 229–233.

[FPP] H. de Fraysseix, J. Pach and R. Pollack, Small sets supporting Fary Embeddings of Planar Graphs, in *Proc. STOC 1988*, ACM, 426–433.

[G] B. Grünbaum, *Convex polytopes*, Springer, Berlin, 1995.

[HZ] M. Hachimori and G. M. Ziegler, Decomposition of simplicial balls and spheres with knots consisting of few edges, *Math. Z.* **235** (2000), 159–171.

[KZ] V. Kaibel, G. Ziegler, Counting lattice triangulations, in *Surveys in combinatorics*, 277–307, London Math. Soc. Lecture Note Ser., **307**, Cambridge Univ. Press, Cambridge, 2003.

[L] L. Lovász, *Geometric representations of graphs*, lecture notes; available electronically at `http://www.cs.elte.hu/~lovasz/geomrep.pdf`.

[K] E. R. van Kampen, Remark on the address of S. S. Cairns, in *Lectures in Topology*, 311–313, University of Michigan Press, Ann Arbor, MI, 1941.

[P] I. Pak, *Lectures on discrete and polyhedral geometry*, monograph draft; available electronically at `http://www.math.ucla.edu/~pak/book.htm`.

[RRS] A. Ribó Mor, G. Rote and A. Schulz, Small grid embeddings of 3-polytopes, *Discrete Comput. Geom.* **45** (2011), 65–87.

[R] J. Richter-Gebert, *Realization spaces of polytopes*, *Lecture Notes in Math.*, Springer, 1996.

[Ro] G. Rote, The number of spanning trees in a planar graph, in *Oberwolfach Reports* vol. 2, EMS, 2005, 969–973.

[RZ] J. Richter-Gebert and G. M. Ziegler, Realization spaces of 4-polytopes are universal, *Bull. Amer. Math. Soc.* **32** (1995), 403–412.

[OS] S. Onn and B. Sturmfels, A quantitative Steinitz' theorem, *Beiträge zur Algebra und Geometrie* **35** (1994), 125–129.

[T1] W. T. Tutte, Convex representations of graphs, *Proc. Lond. Math. Soc.* **10** (1960), 304–320.

[T2] W. T. Tutte, How to draw a graph, *Proc. Lond. Math. Soc.* **13** (1963), 743–768.

[W] E. Welzl, The number of triangulations on planar point sets, in *Lecture Notes in Comput. Sci.* **4372**, Springer, Berlin, 2007, 1–4.

[Wi] N. Witte, A vertex decomposable 3-ball and 3-sphere with a knot consisting of 6 edges, *Electronic Geometry Model* No. 2001.05.003, available at `http://tinyurl.com/7bxdoxp`

[Wo] R. Woodroofe, Vertex decomposable graphs and obstructions to shellability, *Proc. Amer. Math. Soc.* **10** (2009), 3235–3234.

[Z1] G. M. Ziegler, *Lectures on polytopes*, Springer, New York, 1995.

[Z2] G. M. Ziegler, Convex polytopes: extremal constructions and f-vector shapes, in *Geometric Combinatorics*, AMS, Providence, RI, 2007, 617–691; `arXiv: math.MG/0411400`.