Sol.) Let \( D \) be the unit disc, upward pointing normal so that \( \partial S = \partial D \):

\[
\int_S \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{A} \cdot d\mathbf{S} = \oint_{\partial D} \mathbf{A} \cdot d\mathbf{S} = \int_S \mathbf{F} \cdot d\mathbf{S} = \int_D \mathbf{F}(x,y,0) \cdot \mathbf{n} dS = \int_D \mathbf{F} \cdot \mathbf{n} dS = \oint_{\partial D} = \pi
\]

**Remark (orientation and normal vectors):**

- An orientation, denoted as \( \mathbf{e}_p \), on a surface \( S \) is a choice of unit normal vector field, i.e. \( \| \mathbf{e}_p \| = 1 \) and \( \mathbf{e}_p \) is normal to \( S \) at any point \( P \). Then
  \[
  \int_S \mathbf{F} \cdot d\mathbf{S} \overset{\text{def}}{=} \int_S \mathbf{F} \cdot \mathbf{e}_p \cdot d\mathbf{S}.
  \]

- When a parametrization \( \mathbf{G} : D \to S \) is given, it induces
  \[
  \mathbf{n} = \mathbf{T}_u \times \mathbf{T}_v.
  \]
  This vector is also normal to \( S \), so if \( \mathbf{G} \) is regular, \( \mathbf{n}/\| \mathbf{n} \| \) gives an orientation on \( S \).

- If \( \mathbf{n}/\| \mathbf{n} \| = \mathbf{e}_p \), then
  \[
  d\mathbf{S} = \mathbf{e}_p dS = \frac{\partial \mathbf{n}}{\partial u} \partial u dudv = \mathbf{n} dudv.
  \]
  But still, \( \mathbf{n} \) need not be an orientation itself, since \( \mathbf{n} \) may not be unit normal in general.
Ex. #21 \( \mathbf{F} = \langle y^2, x^3, z^3 \rangle \). Show that if \( C_1, C_2 \) are given by

Then
\[
\oint_{C_1} \mathbf{F} \cdot d\mathbf{s} = \oint_{C_2} \mathbf{F} \cdot d\mathbf{s}.
\]

So \( \text{curl} (\mathbf{F}) = \langle 0, 0, 2x-2y \rangle \). So if \( \mathbf{e}_p \) denotes the orientation of the cylinder, then \( \mathbf{e}_p = \langle \cos \theta, \sin \theta, 0 \rangle \) and
\[\text{curl} (\mathbf{F}) \cdot \mathbf{e}_p = 0.\]

Now let \( S \) be the part of cylinder between \( C_1 \) and \( C_2 \). Then

and \( ds = C_1 - C_2 \). So we have
\[
\oint_{C_1} \mathbf{F} \cdot d\mathbf{s} - \oint_{C_2} \mathbf{F} \cdot d\mathbf{s} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{s}
\]

\[= \int_S \text{curl} (\mathbf{F}) \cdot d\mathbf{S},\]

\[= \int_S \text{curl} (\mathbf{F}) \cdot \mathbf{e}_p \ d\mathbf{S} = 0.\]
Section 18.1. Green's Theorem

Summary

1. Given a simple closed curve, we equip it with the CCW orientation unless stated otherwise.

2. For a planar region \( D \), we regard it as an oriented surface in the \( xy \)-plane with the orientation \( \mathbf{k} \). Then the boundary orientation for \( \partial D \) is determined the same as in the Stokes' Thm case.

3. (Green Thm) For \( \mathbf{F} = \langle F_1, F_2 \rangle \) differentiable on \( D \),
   \[
   \oint_{\partial D} F_1 \, dx + F_2 \, dy = \iint_D \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \, dx \, dy.
   \]

Ex #3
Evaluate \( \oint_C y^3 \, dx + x^2 \, dy \), \( C \) is the unit square \([0,1] \times [0,1] \), CCW.

Sol)
By Green's Thm, we have
\[
\oint_C y^3 \, dx + x^2 \, dy = \iint_D \left( \frac{\partial x^2}{\partial x} - \frac{\partial y^3}{\partial y} \right) \, dx \, dy
= \iint_D 2(x-y) \, dx \, dy = 0. \]

Ex #12
Calculate \( \oint_C x^3 \, dx + 4x \, dy \) along \( C \) given by

Sol)
We want to use the Green's Thm. To this end, we have to form a closed curve.
Let \( BA \) be the line segment from \( B \) to \( A \) (and oriented in this direction).
Then $C + \overline{BA}$ is
1. closed,
2. $\text{CCW}$.

Let $D$ be the region enclosed by $C + \overline{BA}$. Then
\[
\int_{C} x^3 \, dx + 4x \, dy + \int_{\overline{BA}} x^3 \, dx + 4x \, dy
\]
\[
= \int_{D} \frac{\partial}{\partial y}(x^3) - \frac{\partial}{\partial x}(4x) \, dxdy = \int_{D} 4 \, dxdy.
\]
On the other hand, $\overline{BA}$ is parametrized as $\overline{C}(t) = \langle -1, -t + t \rangle$, so
\[
\int_{\overline{BA}} x^3 \, dx + 4x \, dy = \int_{0}^{1} -4 \, dt = -4.
\]
Therefore
\[
\int_{C} x^3 \, dx + 4x \, dy = 4 + 4 \text{Area} (D).
\]

---

**Ex#17.**

Find the area of the region $D$ : between $x$- axis and cycloid $C(t) = (t - \sin(t), 1 - \cos(t)), 0 \leq t \leq 2\pi$.

**Sol.**

We give positive $x$-orientation for convenience.

**Observations**

1. $C : C(t) = (t - \sin(t), 1 - \cos(t))$ traces $C$ in $\text{CW}$ direction as a part of $\partial D$.
2. **Green Theorem** says

   \[
   \text{Area} (D) = \int_{\partial D} y \, dx = \int_{\partial D} x \, dy = \frac{1}{2} \int_{\partial D} x \, dy - y \, dx.
   \]

   We choose the identity
   \[
   \text{Area} (D) = \int_{\partial D} -y \, dx
   \]
   for the following region
1. $y = 0$ on $L$, so $\int_{L} -y \, dx = 0$.
2. By the observation, $\partial D = L - C$.

   $\Rightarrow \int_{\partial D} = \int_{L} - \int_{C}$.
Then we have
\[ \text{Area}(D) = \oint_{D} y \, dx = \int_{C_3} y \, dx - \int_{C_1} y \, dx \]
\[ = \int_{C} y \, dx = \int_{0}^{2\pi} (1 - \cos t) \, d(t - \sin t) \]
\[ = \int_{0}^{2\pi} (1 - \cos t)^2 \, dt = 3\pi. \]

\[ \sqrt{\text{Ex \# 23}} \]

**C_R:** \( x^2 + y^2 = R^2. \) Calculate \( \oint_{C_3} \mathbf{F} \cdot ds \) assuming

\( 1 \oint_{C_1} \mathbf{F} \cdot ds = 9 \) \hspace{1cm} \text{AND} \hspace{1cm} \( \frac{\partial F}{\partial y} - \frac{\partial F}{\partial x} = x^2 + y^2 \) on \( 1 \leq x^2 + y^2 \leq 4. \)

**Sol:** The situation is as follows: Let \( D : 1 \leq x^2 + y^2 \leq 4. \) Then

\( \partial D = C_2 - C_1 \)

So by the Green's Theorem,

\[ \oint_{C_2} \mathbf{F} \cdot ds - \oint_{C_1} \mathbf{F} \cdot ds = \oint_{C_2 - C_1} \mathbf{F} \cdot ds \]
\[ = \int_{D} \left( \frac{\partial F}{\partial x} - \frac{\partial F}{\partial y} \right) \, dx \, dy \]
\[ = 9 \] by \( 1 \)
\[ = \int_{D} (x^2 + y^2) \, dx \, dy \]
\[ \text{by } \partial \]

This gives
\[ \oint_{C_2} \mathbf{F} \cdot ds = 9 + \int_{0}^{2\pi} \int_{1}^{\sqrt{4}} (x^2 + y^2) \, dr \, d\theta \]
\[ = 9 + 2\pi \cdot \frac{15\pi}{4} = 9 + \frac{15\pi}{2}. \]
Ex. #32

(a) \( C: \) line segment from \((x_1, y_1)\) to \((x_2, y_2)\). Show that

\[
\frac{1}{2} \int_C y \, dx + x \, dy = \frac{1}{2} (x_1 y_2 - x_2 y_1).
\]

(b) Show that the area of the polygon \( D \) with vertices \( A_1 = (x_1, y_1), \ldots, A_n = (x_n, y_n) \) (in \( \text{CCW} \) order) is

\[
\frac{1}{2} \sum_{i=1}^{n} (x_i y_{i+1} - x_{i+1} y_i),
\]

where \((x_n, y_n) = (x_1, y_1)\).

Sol)

(a) \( C(t) = \langle (1-t)x_1 + tx_2, (1-t)y_1 + ty_2 \rangle, \ 0 \leq t \leq 1 \) parametrizes \( C \).

So

\[
\frac{1}{2} \int_C y \, dx + x \, dy = \frac{1}{2} \int_0^1 \left( (1-t)y_1 + ty_2 \right) \, dx_2 - (x_2 - x_1) \, dt
+ \frac{1}{2} \int_0^1 \left( (1-t)x_1 + tx_2 \right) \, dy_2 - (y_2 - y_1) \, dt
\]

\[
= \frac{1}{2} \left[ \left\{ \frac{1}{2} y_1 + \frac{1}{2} y_2 \right\} (x_2 - x_1) + \left\{ \frac{1}{2} x_1 + \frac{1}{2} x_2 \right\} (y_2 - y_1) \right]
\]

\[
= \frac{1}{2} (x_1 y_2 - x_2 y_1).
\]

(b) \( \text{Area}(D) = \sum_{i=1}^{n} \frac{1}{2} \int_{A_iA_{i+1}} y \, dx + x \, dy
\]

\[
= \sum_{i=1}^{n} \frac{1}{2} \int_{A_iA_{i+1}} (x_i y_{i+1} - x_{i+1} y_i)\)
\]

\[
= \sum_{i=1}^{n-1} \frac{1}{2} (x_i y_{i+1} - x_{i+1} y_i).
\]