• Or more directly, since \( z=0 \) on \( C_i \),

\[
I = \oint_C (\sin x + y) \, dx + 0 \, dy + \sin y \, dz
\]

\[
= \oint_C (\sin x + y) \, dx
\]

\[
= \oint_C d(-\cos x) + \oint_C y \, dx.
\]

\[
= 0 \quad \text{by FTC.} \quad \text{As in the previous method...}
\]

\[\text{Summary}\]

• Conservative \( \Rightarrow \) Cross-Partial Condition (CPC)

• CPC + simply connected domain \( \Rightarrow \) conservative.

\[\text{Ex \#14}\]

Find a potential function for \( \mathbf{F} = \langle e^x(z+1), -\cos y, e^x \rangle \), or determine that \( \mathbf{F} \) is not conservative.

\[\text{Sol}\]

• CPC is satisfied: \( \frac{\partial F_x}{\partial x} = 0 = \frac{\partial F_y}{\partial y}, \quad \frac{\partial F_y}{\partial z} = 0 = \frac{\partial F_z}{\partial z}, \quad \frac{\partial F_y}{\partial z} = e^x = \frac{\partial F_z}{\partial z} \).

• The domain of \( \mathbf{F} \) is \( \mathbb{R}^3 \), which is simply connected.

\[\Rightarrow \exists \text{ potential function } V.\]

• Let \( V_1 = \int F_1 \, dx = e^x(z+1) + C \) with the choice \( C=0 \). Then

\( \nabla V_1 = \langle e^x(z+1), 0, e^x \rangle \) so

\( \nabla (V-V_1) = \mathbf{F} - \nabla V_1 = \langle 0, -\cos y, 0 \rangle.\)

• Let \( V_2 = \int -\cos y \, dy = \sin y + C \). Then \( \nabla V_2 = \langle 0, -\cos y, 0 \rangle \) and

\( \nabla (V-V_1-V_2) = 0 \quad \Rightarrow \quad V_1 + V_2 \) is a potential function for \( \mathbf{F}. \)
Section 17.4: Parametrized Surfaces and Surface Integrals.

Overview

- **Single Integral**
  \[ \int_a^b f(x) \, dx \]

- **Multiple Integral**
  \[ \int_D f(x,y) \, dx \, dy \]
  - much harder to describe the domain \( D \)

- **Line Integral**
  \[ \int_C f \, ds \quad \text{or} \quad \int_C F \cdot ds \]
  - domain is now 'warped', so need to parametrize it.
  - distinction between scalar and vector appears
  - orientation matters

- **Surface Integral**
  \[ \int_S f \, dS \quad \text{or} \quad \int_S F \cdot n \, dS \]
  - those characteristics are combined.

Summary

- A parametrized surface \([\text{surface } S] + [\text{parametrization } G: D \to S]\)
- Points in \( S \) are written as \((x,y,z) = (x(u,v), y(u,v), z(u,v)) = G(u,v)\).
- \( u,v \) are called parameters, \( D \) parameter domain

\[ G(u,v) \]
- \( T_u = \frac{\partial G}{\partial u} \), \( T_v = \frac{\partial G}{\partial v} \) are called tangent vectors.
- \( n = T_u \times T_v \) is called a normal vector.
- \( G \) is called regular at \( P \) if \( n(P) \neq 0 \).
Goal

How to parametrize a surface?

- Cylinder: \( G(\theta, z) = (R \cos \theta, R \sin \theta, z) \)
- Sphere: \( G(\phi, \theta) = (R \cos \theta \sin \phi, R \sin \theta \sin \phi, R \cos \phi) \)
- Graph: \( G(x, y) = (x, y, f(x, y)) \).

But of course different parametrizations are also possible.

Ex

Parametrize the surface \( S \): part of \( z = 1 - x^2 - y^2 \) on \( z \geq 0 \).

Sol) 1

\[ z = 1 - x^2 - y^2 \] Intersects \( z = 0 \) on \( x^2 + y^2 = 1 \).

\[ \Rightarrow S \text{ : graph of } z = 1 - x^2 - y^2 \text{ on } D : x^2 + y^2 \leq 1. \]

\[ \Rightarrow G(x, y) = (x, y, 1 - x^2 - y^2) \text{ on } D. \]

2

Using cylindrical coordinates,

\[ G(r, \theta) = (r \cos \theta, r \sin \theta, 1 - r^2) \text{ on } D : 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi. \]

Q: Which is better? A: It depends on situation.

Ex

Parametrize the part of the plane \( \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \), where \( x, y, z \geq 0 \).

Sol)

NOTE \( S \) is a part of the graph

\[ z = c \left( 1 - \frac{x}{a} - \frac{y}{b} \right) \]

on the triangular domain \( D \) with vertices \( (0, 0), (a, 0), (0, b) \).

\[ \Rightarrow G(x, y, z) = (x, y, c(1 - \frac{x}{a} - \frac{y}{b})) \text{, } (x, y) \in D. \]