Chapter 16.3 Triple Integral

Ex#13 Evaluate \( \iiint_W e^z \, dV \), where \( x+y+z \leq 1 \) and \( x, y, z \geq 0 \).

Set) \( W \) looks like:

(1) Find an expression for \( W \) (i.e. describe \( W \)). Basically, we want to write down \( W \) as a **simple region**.

- \( W \) is simple in each direction! We thus want to express \( W \) as simple in the \( z \)-direction, i.e.,

  \[ W : (x,y) \in D \quad \text{and} \quad 0 \leq z \leq 1 \]

- \( D \) is the projection of \( W \) onto the \( xy \)-plane. It is the triangle with vertices \((0,0), (0,1), (1,0)\).

  \[ \Rightarrow D : 0 \leq x \leq 1 \quad \text{and} \quad 0 \leq y \leq 1-x. \]

(2) We want to find out the bounds for \( z \):

\[ \begin{cases} &0 \leq z \quad \text{is clear}, \\ &x+y+z \leq 1 \Rightarrow z \leq 1-x-y. \end{cases} \Rightarrow 0 \leq z \leq 1-x-y. \]

(2) Evaluate the integral: \( \iiint_W e^z \, dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} e^z \, dz \, dy \, dx \)

\[ = \int_0^1 \int_0^{1-x} (e^{x+y} - 1) \, dy \, dx \]

\[ = \ldots \]
[5x23] Evaluate \( \iiint_W xz \, dV \), where \( W \) is bounded by \( \begin{cases} \frac{x^2}{4} + \frac{y^2}{9} = 1, \\
x^2 + y^2 + z^2 = 16. \\ \text{in the 1st octant : } x, y, z \geq 0. \end{cases} \)

S ol.) The situation is as follows:

\[ x^2 + y^2 + z^2 = 16, \]
\[ \frac{x^2}{4} + \frac{y^2}{9} = 1. \]

1. Find an expression for \( W \):
   1. \( W \) is simple in \( z \)-direction:
      \[ W : (x, y) \in D \text{ and } 0 \leq z \leq \square \]

2. \( D \) is the projection of \( W \) onto the \( xy \)-plane. It is clearly
   \[ D : x \geq 0, \ y \geq 0, \ \frac{x^2}{4} + \frac{y^2}{9} \leq 1. \]
   (This is hard to be read out from the equations that define \( W \),
   sketching the region is VERY important!)

3. Bounds for \( z \):
   \[ z \geq 0 \text{ is clear}. \]
   \[ - \text{ The top of } W \text{ is part of the sphere } x^2 + y^2 + z^2 = 16. \]
   This gives
   \[ z \leq \sqrt{16 - x^2 - y^2} \]
   \[ \Rightarrow 0 \leq z \leq \sqrt{16 - x^2 - y^2}. \]

(2) The integral is now written as
\[ \iiint_W xz \, dV = \int_0^2 \int_0^2 \int_0^{\sqrt{16 - x^2 - y^2}} xz \, dz \, dy \, dx. \]
Section 16.5. Application of Multiple Integrals.

Keywords:
- Total mass: \( M = \iint_D \rho \, dA \)
- Moments: \( M_x, M_y = \iint_D x \rho \, dA, \iint_D y \rho \, dA \)
- CoM: \( \bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M} \).
- Moment of Inertia: \( I_x, I_y = \iint_D y^2 \rho \, dA, \iint_D x^2 \rho \, dA \).

Ex #11.

Find the centroid of the region bounded by \( y = 1 - x^2 \) and \( y = 0 \).

Let \( D \) denote the region.

(1) Find the area of \( D \):
\[
M = A = \int_{-1}^{1} (1 - x^2) \, dx = \frac{4}{3}.
\]
\( \therefore \rho = 1 \) uniformly.

(2) Find the moments of \( D \):
- \( M_y = \int_{-1}^{1} \int_{0}^{1 - x^2} x \, dy \, dx = 0 \), or by symmetry.
- \( M_x = \int_{-1}^{1} \int_{0}^{1 - x^2} y \, dy \, dx = \int_{-1}^{1} \frac{1}{3} (1 - x^2)^2 \, dx = \frac{8}{15} \).

(3) Calculate the centroid:
\[
\bar{x} = \frac{M_y}{M} = 0, \quad \bar{y} = \frac{M_x}{M} = \frac{\frac{8}{15}}{\frac{4}{3}} = \frac{2}{5}.
\]
Ex #25

Find the CoM of the region \( |x|+|y| \leq 1 \) with density \( p(x,y) = x^2 \).

Let \( D \) denote the region.

1. Find an expression for \( D \):
   - How \( D \) look like?

Note that if \( |x|+|y| \leq 1 \), then:
   - \( -x+1+y \leq 1 \), \( x+1-y \leq 1 \), \( -x+1-y \leq 1 \).

\( \Rightarrow D \) is symmetric with respect to \( x, y \)-axes!

\( \Rightarrow D \) is completely determined by its shape in the 1st quadrant.

But when \( x \geq 0, y \geq 0 \), \( 1 \geq |x|+|y| = x+y \) and

\( \Rightarrow D \) looks like

(2) Calculate the total mass \( M \):

\[
M = \iiint_D p \, dA = \int_{-1}^{1} \int_{-1-x}^{1-x} p(x,y) \, dy \, dx + \int_{0}^{1} \int_{-1}^{1-x} p(x,y) \, dy \, dx
\]

\[
= \int_{1}^{0} 2(x+1)x^2 \, dx + \int_{0}^{1} 2(1-x) x^2 \, dx = \frac{1}{3}.
\]
(3) Calculate the moments,

\[ M_y = \int_D x \rho \, dA = \int_{-1}^1 \int_{-1}^{x+1} x \rho(x,y) \, dy \, dx + \int_{-1}^{1-x} \int_{x-1}^0 x \rho(x,y) \, dy \, dx \]

\[ = \int_{-1}^0 2(x+1)x^2 \, dx + \int_{0}^1 2(1-x)x^2 \, dx = 0, \]

\[ M_x = \int_D y \rho \, dA = \int_{-1}^1 \int_{-1}^{x+1} y x^2 \, dy \, dx + \int_{-1}^{1-x} \int_{x-1}^0 y x^2 \, dy \, dx = 0, \]

or by symmetry, we get the same conclusion.

(4) Calculate the CoM:

\[ x_{CM} = \frac{M_y}{M} = 0, \quad y_{CM} = \frac{M_x}{M} = 0. \]