Section 9.6. Improper Integrals

Summary. An improper integral is defined by taking limits to bounds: for \(-\infty < a < b < \infty\),

\[
\int_a^b f(x) \, dx = \lim_{S \to a^+} \int_S^b f(x) \, dx.
\]

We say \(\int_a^b f(x) \, dx\) converges if this limit converges, diverges if it diverges.

- (p-integral) For \(a > 0\),

<table>
<thead>
<tr>
<th>(\int_a^\infty \frac{dx}{x^p})</th>
<th>(\int_0^a \frac{dx}{x^p})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p &gt; 1)</td>
<td>(\int_a^\infty \frac{dx}{x^p}) converges (\infty)</td>
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<tr>
<td>(p = 1)</td>
<td>(\int_0^a \frac{dx}{x}) diverges (\infty)</td>
</tr>
<tr>
<td>(p &lt; 1)</td>
<td>(\int_0^a \frac{dx}{x^p}) converges (\infty)</td>
</tr>
</tbody>
</table>

- (Comparison test) If \(f(x) \geq g(x) > 0\), then

\[
\int_a^b f(x) \, dx \geq \int_a^b g(x) \, dx > 0.
\]

Consequently:

1. If \(\int_a^b f(x) \, dx < \infty\) \(\Rightarrow\) \(\int_a^b g(x) \, dx < \infty\)
2. If \(\int_a^b g(x) \, dx = \infty\) \(\Rightarrow\) \(\int_a^b f(x) \, dx = \infty\).

- Inconclusive if \(\int_a^b f(x) \, dx = \infty\) or \(\int_a^b g(x) \, dx < \infty\).

**Ex #4**

Compute \(\lim_{R \to \infty} \int_0^R \frac{dx}{(3-x)^{3/2}}\) to determine whether \(\int_0^3 \frac{dx}{(3-x)^{3/2}}\) converges.

\[
\int_0^R \frac{dx}{(3-x)^{3/2}} = \int_0^R \frac{2}{\sqrt{3-x}} \, dx = \frac{2}{\sqrt{3-R}} - \frac{2}{\sqrt{3}}.
\]

Taking \(R \to \infty\), \(\frac{2}{\sqrt{3-R}} \to 0\) and therefore \(\int_0^3 \frac{dx}{(3-x)^{3/2}}\) diverges.

**Ex #16**

Investigate \(\int_0^\infty e^{-2x} \, dx\).

\[
\int_0^\infty e^{-2x} \, dx = -\frac{1}{2} e^{-2x} \bigg|_0^\infty = \frac{1}{2} (e^0 - e^{-\infty}) \to \frac{1}{2} e^0 \quad \therefore \quad \text{converges}.
\]
Ex #22: Investigate \( \int_1^2 \frac{dx}{(x-1)^2} \).

Sol. \( \int_1^2 \frac{dx}{(x-1)^2} = \int_0^1 \frac{du}{u^2} \quad (x-1 = u) \) diverges by \( p \)-test.

Ex #27: Investigate \( \int_0^\infty \frac{x}{(1+x^3)^3} \, dx \).

Sol. 1.) \( \int_0^R \frac{x}{(1+x^3)^3} \, dx = -\frac{1}{2(1+R^3)} \left| \right|_0^R = \frac{1}{2} \left( 1 - \frac{1}{1+R^3} \right) \to \frac{1}{2} \quad \therefore \) converges

Method 2.) Write

\[
\int_0^\infty \frac{x}{(1+x^3)^3} \, dx = \int_0^1 \frac{x}{(1+x^3)^3} \, dx + \int_1^\infty \frac{x}{(1+x^3)^3} \, dx.
\]

Then \( x^3 + 1 \geq x^3 \Rightarrow \frac{1}{x^3 + 1} \leq \frac{1}{x^3} \Rightarrow \frac{x}{(x^3 + 1)^3} \leq \frac{x}{x^3} \) and

\[
\int_1^\infty \frac{x}{(1+x^3)^3} \, dx \leq \int_1^\infty \frac{dx}{x^3} < \infty
\]

by \( p \)-test. \( \therefore \int_0^\infty \frac{x}{(1+x^3)^3} \, dx \) converges.

Ex #33: Investigate \( \int_1^\infty \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx \).

Sol. \( \frac{e^{\sqrt{x}}}{\sqrt{x}} \geq \frac{1}{\sqrt{x}} \) for \( x > 0 \) and thus \( \int_1^\infty \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx \geq \int_1^\infty \frac{dx}{\sqrt{x}} = \infty \) by \( p \)-integral. \( \therefore \) diverges.

Ex #38: Investigate \( \int_1^2 \frac{dx}{x \ln x} \).

Sol. \( \int_1^2 \frac{dx}{x \ln x} = \ln |x| \bigg|_1^2 = \ln 2 - \ln 1 \to \infty \quad \therefore \) diverges.

Ex #45: Investigate \( \int_{-\infty}^\infty \frac{x}{x^2+1} \, dx \).

Sol. \( \int_{-\infty}^\infty \frac{x}{x^2+1} \, dx = \frac{1}{2} \ln (x^2 + 1) \bigg|_R = \frac{1}{2} \ln \left( \frac{R^2 + 1}{R^2 + 1} \right) \)

does not converge as \( R \to \infty \).

\( \int_0^\infty \frac{x}{x^2+1} \, dx = \frac{1}{2} \ln (x^2 + 1) \bigg|_0^\infty = \infty \quad \therefore \) diverges.

\( \int_0^\infty \frac{x}{x^2+1} \, dx = \int_0^1 \frac{x}{x^2+1} \, dx + \int_1^\infty \frac{x}{x^2+1} \, dx. \) Now note that \( x^2 + 1 \leq 2x^2 \) for \( x \geq 1 \)
implies \( \frac{x}{x^2+1} \geq \frac{1}{2x} \). Then \( \int_1^\infty \frac{x}{x^2+1} \, dx \geq \int_1^\infty \frac{dx}{2x} = \infty \) by \( p \)-integral. \( \therefore \) diverges.

Remark: Although it is an integral of odd function on a symmetric interval, it is simply not zero because it diverges.
Ex #54
Show that \( \int_{1}^{\infty} \frac{dx}{x^{4}+1} \) converges.

\[
x^{4}+1 \geq x^{4} \text{ on } x \geq 1 \Rightarrow \frac{1}{x^{4}} \geq \frac{1}{x^{4}+1} \Rightarrow \frac{1}{x} \geq \frac{1}{\sqrt{x^{4}+1}} \geq 0 \text{ and thus}
\]

\( \int_{1}^{\infty} \frac{dx}{x^{2}} \leq \int_{1}^{\infty} \frac{dx}{x^{4}+1} < \infty \text{ by p-integral.} \)

\( \Rightarrow \) \( \text{converges!} \)

Ex #65
Determine whether \( \int_{1}^{\infty} e^{-(x+\frac{1}{x})} \) converges.

\[
0 \leq e^{-(x+\frac{1}{x})} \leq e^{-x} \Rightarrow \int_{1}^{\infty} e^{-(x+\frac{1}{x})} \, dx \leq \int_{1}^{\infty} e^{-x} \, dx < \infty
\]

and thus it converges.

Ex #69

\( \int_{0}^{1} \frac{dx}{x^{4}+\sqrt{x}} \)

\[
x^{4}+\sqrt{x} \geq \sqrt{x} \Rightarrow \frac{1}{\sqrt{x}} \geq \frac{1}{x^{4}+\sqrt{x}} \geq 0
\]

\( \Rightarrow \int_{0}^{1} \frac{dx}{x^{4}+\sqrt{x}} \leq \int_{0}^{1} \frac{dx}{\sqrt{x}} < \infty. \)

\( \Rightarrow \) \( \text{converges!} \)

Ex #72

\( \int_{0}^{\infty} \frac{dx}{(8x^{2}+x^{4})^{1/3}} \)

- Near \( x = 0 \), \( (8x^{2}+x^{4})^{1/3} \sim 2x^{2/3} \Rightarrow \frac{1}{(8x^{2}+x^{4})^{1/3}} \sim \frac{1}{2x^{2/3}} \).

Since \( \int_{0}^{1} \frac{dx}{2x^{2/3}} \) converges, so does \( \int_{0}^{1} \frac{dx}{(8x^{2}+x^{4})^{1/3}} \).

- Near \( x = \infty \), \( (8x^{2}+x^{4})^{1/3} \sim x^{4/3} \text{ as } x \to \infty. \Rightarrow \frac{1}{(8x^{2}+x^{4})^{1/3}} \sim \frac{1}{x^{4/3}} \).

Since \( \int_{1}^{\infty} \frac{dx}{x^{4/3}} \) converges, so does \( \int_{1}^{\infty} \frac{dx}{(8x^{2}+x^{4})^{1/3}} \).

Therefore, \( \int_{0}^{\infty} \frac{dx}{(8x^{2}+x^{4})^{1/3}} \) converges too!

Remark: (Limit Comparison Test) Suppose \( f(x), g(x) > 0 \) and \( f(x) \sim g(x) \text{ as } x \to \infty. \)

Then \( \int_{0}^{\infty} f(x) \, dx \) and \( \int_{0}^{\infty} g(x) \, dx \) both converge or diverge simultaneously.