Section 8.5. The method of Partial Fraction

Summary (Skipped, see the previous note.)

Ex #9
\[ \int \frac{dx}{(x-2)(x-4)} \]

So 1) It is convenient to remember the following special case:
\[ \frac{1}{(x-a)(x-b)} = \frac{1}{b-a} \left( \frac{1}{x-a} - \frac{1}{x-b} \right) (a+b). \]

So in this case,
\[ \int \frac{dx}{(x-2)(x-4)} = \frac{1}{2} \int \left( \frac{1}{x-4} - \frac{1}{x-2} \right) \, dx \]
\[ = \frac{1}{2} \left( \ln|x-4| - \ln|x-2| \right) + C \]
\[ = \frac{1}{2} \ln \frac{x-4}{x-2} + C. \]

Ex #16
\[ \int \frac{dx}{(x+1)(x+4)}. \]

So 1) Importance of factoring!

Factoring the denominator, \( (x+1)(x^2+x) = x(x+1)^2 \). So we set up
\[ \frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \]

and by clearing denominators and comparing coefficients, we get
\[ \begin{cases} A+B = 0 \\ 2A + B + C = 0 \\ A = 1 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = -1 \\ C = -1 \end{cases}. \]

This gives
\[ \int \frac{dx}{x(x+1)^2} = \int \left( \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2} \right) \, dx \]
\[ = \ln|x| - \ln|x+1| + \frac{1}{x+1} + C. \]

Ex #43
\[ \int \frac{25}{x(x^2+2x+5)} \, dx \]

So 1) Since \( x^2+2x+5 \) has no real root, we set up
\[ \frac{25}{x(x^2+2x+5)} = \frac{A}{x} + \frac{B_1x+C_1}{x^2+2x+5} + \frac{B_2x+C_2}{(x^2+2x+5)^2}. \]
Using the same trick as before, we get

\[ A = 1, \quad B_1 = -1, \quad C_1 = -2, \quad B_2 = -5, \quad C = -10. \]

So

\[
\int \frac{2x\,dx}{x(x^2+2x+5)^2} = \int \frac{dx}{x} - \int \left( \frac{x+2}{x^2+2x+5} + \frac{5x+10}{(x^2+2x+5)^2} \right) \, dx.
\]

\[ = \ln|x| - I. \]

To evaluate \( I \), use \( x+1 = 2 \tan \theta \). Then \( x^2+2x+5 = 4 \sec^2 \theta \) and

\[ I = \int \left( \frac{1}{2} (2 \tan \theta + 1) + \frac{5}{8} \cos \theta \left( 2 \tan \theta + 1 \right) \right) \, d\theta. \]

\[ = \int \left( \sin \theta + \frac{5}{16} \left( 2 \cos \theta \sin \theta + \cos \theta \right) \right) \, d\theta \]

\[ = \ln |\sec \theta| + \frac{\theta}{2} - \frac{5}{8} \cos^2 \theta + \frac{5}{16} \left( \sin \theta \cos \theta + \theta \right) + C \]

\[ = \ln \sqrt{1 + \tan^2 \theta} + \frac{13}{16} \theta - \frac{5}{8} \frac{1}{1 + \tan^2 \theta} + \frac{5}{16} \frac{\tan \theta}{1 + \tan^2 \theta} + C \]

\[ = \ln \sqrt{x^2+2x+5} + \frac{13}{16} \arctan \left( \frac{x+1}{2} \right) - \frac{5}{2} \frac{1}{x^2+2x+5} + \frac{5}{8} \frac{x+1}{x^2+2x+5} + C. \]

\[ \boxed{\text{III}} \]
Ex #50

Evaluate \( \int \frac{dy}{x^{1/2} - x^{1/3}} \)

\[ \int \frac{dx}{x^{1/2} - x^{1/3}} = \int \frac{dx}{x^{1/3} (x^{1/6} - 1)} \quad \text{We apply } u = x^{1/6} \iff x = u^6, \quad dx = 6u^5 \, du. \]

\[ = \int \frac{6u^5}{u^2 (u-1)} \, du = \int \frac{6u^3}{u-1} \, du = 6 \int (u^2 + u + \frac{1}{u-1}) \, du \]

\[ = 6 \left( \frac{u^3}{3} + \frac{u^2}{2} + u + \ln|u-1| \right) + C \]

\[ = 2x^{1/2} + 3x^{1/3} + 6x^{1/6} + \ln|x^{1/6}-1| + C \]

Ex #42

\[ \int \frac{9 \, dx}{(x+1)(x^2-2x+6)} \]

By the method of partial fraction, we can set up

\[ \frac{9}{(x+1)(x^2-2x+6)} = \frac{A}{x+1} + \frac{Bx+C}{(x-1)^2+5} \]

Then \( 9 = A(x^2-2x+6) + (Bx+C)(x+1) \) and

\[ A+1 = 0 \quad \Rightarrow \quad A = -1 \]

\[ B - 3A + C = 0 \quad \Rightarrow \quad B = -A = 1 \]

\[ 6A + C = 9 \quad \Rightarrow \quad C = 9 - 6A = 9 - 6(-1) = 15 \]

So we have

\[ \int \frac{9 \, dx}{(x+1)(x^2-2x+6)} = \int \frac{dx}{x+1} - \int \frac{x-3}{(x-1)^2+5} \, dx \]

\[ = \ln|x+1| - \left( \frac{x-1}{(x-1)^2 + 5} - \frac{2}{(x-1)^2 + 5} \right) \, dx \]

Using \( u = x^2 - 2x + 6, \quad du = 2(x-1) \, dx \) and

\[ -\int \frac{x-1}{(x-1)^2 + 5} \, dx \]

\[ = -\int \frac{du}{u} = -\frac{1}{2} \ln|u| = -\frac{1}{2} \ln(x^2 - 2x + 6). \]

Using \( x-1 = \sqrt{5} \tan \theta, \)

\[ \int \frac{2}{(x-1)^2 + 5} \, dx = \int \frac{2}{\sqrt{5} \sec^2 \theta} \cdot \sqrt{5} \sec^2 \theta \, d\theta = \frac{2}{\sqrt{5}} \theta + C \]

\[ = \frac{2}{\sqrt{5}} \tan^{-1} \left( \frac{x-1}{\sqrt{5}} \right) + C. \]

Therefore

\[ \int \frac{9 \, dx}{(x+1)(x^2-2x+6)} = \ln|x+1| - \frac{1}{2} \ln(x^2 - 2x + 6) + \frac{2}{\sqrt{5}} \tan^{-1} \left( \frac{x-1}{\sqrt{5}} \right) + C. \]
Ex #63  Calculate \( \int \frac{x^2}{(x^2-1)^{3/2}} \, dx \)

Sol)  Note that (with \( u = x^2 - 1 \))

\[
\int \frac{x}{(x^2-1)^{3/2}} \, dx = \frac{1}{2} \int \frac{d}{u^{1/2}} = -\frac{1}{\sqrt{u}} + C = -\frac{1}{\sqrt{x^2-1}} + C.
\]

So we have

\[
\int \frac{x^2}{(x^2-1)^{3/2}} \, dx = x \cdot \left( -\frac{1}{\sqrt{x^2-1}} \right) + \int \frac{dx}{\sqrt{x^2-1}}
\]

with \( x = \sec \theta \),

\[
= -\frac{x}{\sqrt{x^2-1}} + \int \sec \theta \, d\theta
\]

\[
= -\frac{x}{\sqrt{x^2-1}} + \ln |\sec \theta + \tan \theta| + C
\]

\[
= -\frac{x}{\sqrt{x^2-1}} + \ln |x + \sqrt{x^2-1}| + C.
\]

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Ex #65  \( \int \frac{\sqrt{x}}{x^2+1} \, dx \).

Sol)  Put \( x^{3/2} = \tan \theta \). Then \( \frac{3}{2} \sqrt{x} \, dx = \sec^2 \theta \, d\theta \) and

\[
\int \frac{\sqrt{x}}{x^2+1} \, dx = \int \frac{\frac{3}{2} \sec^2 \theta \, d\theta}{\tan \theta + 1}
\]

\[
= \frac{2}{3} \sec \theta \, d\theta + C
\]

\[
= \frac{2}{3} \arctan (x^{3/2}) + C.
\]
**Section 2.6: Improper Integrals**

**Summary**
- An improper integral is defined by taking limits to ‘bounds’; for \(-\infty \leq a \leq b \leq \infty\),
  \[ \int_{a}^{b} f(x) \, dx = \lim_{R \to \infty} \int_{a}^{b} f(x) \, dx \]

- (P-integral) For \( a > 0 \),
  \[ \int_{a}^{\infty} \frac{dx}{x^p}, \quad \int_{0}^{a} \frac{dx}{x^p} \]
  \[
  \begin{array}{c|c|c}
  p > 1 & \text{Converges} & = \infty \\
  p = 1 & = \infty & = \infty \\
  p < 1 & = \infty & \text{Converges} \\
  \end{array}
  \]

- (Comparison test) If \( f(x) \geq g(x) \geq 0 \) for \( a < x < b \), \(-\infty \leq a \leq b \leq \infty\),
  \[ \int_{a}^{b} f(x) \, dx \geq \int_{a}^{b} g(x) \, dx \geq 0. \]
  Consequently,
  \[
  \begin{align*}
  &\{ \text{If } \int_{a}^{b} f(x) \, dx < \infty, \text{ then } \int_{a}^{b} g(x) \, dx < \infty \\
  &\{ \text{If } \int_{a}^{b} g(x) \, dx = \infty, \text{ then } \int_{a}^{b} f(x) \, dx = \infty.
  \end{align*}
  \]

- (Limit comparison test) If \( f(x), g(x) \geq 0 \) for \( a < x < b \), then
  \[
  \begin{align*}
  &\text{If } \lim_{x \to a^+} \frac{f(x)}{g(x)} < \infty, \text{ lim }_{x \to b^-} \frac{f(x)}{g(x)} < \infty \text{ and } \int_{a}^{b} g(x) \, dx < \infty, \text{ then } \\
  &\int_{a}^{b} f(x) \, dx < \infty.
  \end{align*}
  \]

  \[
  \begin{align*}
  &\text{If } \lim_{x \to a^+} \frac{f(x)}{g(x)} > 0, \text{ lim }_{x \to b^-} \frac{f(x)}{g(x)} > 0 \text{ and } \int_{a}^{b} g(x) \, dx = \infty, \text{ then } \\
  &\int_{a}^{b} f(x) \, dx = \infty.
  \end{align*}
  \]

**Tip** Checking the convergence of improper integrals.

1st way) Apply the definition directly.

2nd way) Investigate the 'singularities' of \( f(x) \).

1. \( x = \pm \infty \) are singularities of \( f \)
2. The points where \( f(x) \to \pm \infty \) are also singularities.
Ex #4: Compute \( \lim_{R \to 3^-} \int_0^R \frac{dx}{(3-x)^{3/2}} \) to determine whether \( \int_0^3 \frac{dx}{(3-x)^{3/2}} \) converges.

So1) \[
\int_0^R \frac{dx}{(3-x)^{3/2}} = \left[ \frac{2}{\sqrt{3-x}} \right]_0^R = \frac{2}{\sqrt{3-R}} - \frac{2}{\sqrt{3}}.
\]
Taking \( R \to 3^- \), \( \frac{2}{\sqrt{3-R}} \to \infty \) and therefore \( \int_0^3 \frac{dx}{(3-x)^{3/2}} = \infty \).

Ex #11: Investigate \( \int_0^4 \frac{dx}{\sqrt{4-x}} \)

So1) \( u = 4-x \) gives \( \int_0^4 \frac{du}{\sqrt{u}} = \int_0^4 \frac{du}{u^{1/2}} \), which converges by p-test. Moreover,
\[
\int_0^4 \frac{du}{u^{1/2}} = 2 \sqrt{u} \bigg|_0^4 = 4
\]

Ex #16: Investigate \( \int_2^\infty e^{-2x} \, dx \).

So1) \[
\int_2^\infty e^{-2x} \, dx = -\frac{1}{2} e^{-2x} \bigg|_2^\infty = \frac{1}{2} (e^{-4} - e^{-R}).
\]
Taking \( R \to \infty \), \( e^{-R} \to 0 \) and hence \( \int_2^\infty e^{-2x} \, dx \) converges and
\[
\int_2^\infty e^{-2x} \, dx = \frac{1}{2} e^{-4}
\]

Ex #22: Investigate \( \int_1^2 \frac{dx}{(x-1)^2} \).

So1) \[
\int_1^2 \frac{dx}{(x-1)^2} = \int_0^1 \frac{du}{u^2} \quad (u = x-1). \quad \text{By p-test, it diverges to } \infty
\]

Ex #27: Investigate \( \int_0^\infty \frac{x}{(1+x^2)^2} \, dx \)

So1) \[
\int_0^\infty \frac{x}{(1+x^2)^2} \, dx = \frac{1}{2} \left( \frac{1}{1+R^2} \right) \bigg|_0^\infty = \frac{1}{2} \left( 1 - \frac{1}{1+R^2} \right) \to \frac{1}{2}.
\]
\[
\frac{x}{(1+x^2)^2} \sim \frac{1}{x^3} \quad \text{as } x \to \infty. \quad \text{So we at least know that } \int_0^\infty \frac{x}{(1+x^2)^2} \, dy \text{ converges by p-test.}
\]

Ex #36: Investigate \( \int_0^{\pi/2} \tan x \, dx \).

So1) \[
\int_0^{\pi/2} \tan x \, dx = \ln(\sec x) \bigg|_0^{\pi/2} = \ln(\sec R) \to \infty \quad \text{as } R \to \frac{\pi}{2}.
\]
So it diverges to \( \infty \).