Section 7.1. Derivative of \( f(x) = b^x \) and the Number e.

Ex #77. Evaluate \( \int_0^1 xe^{-x^2} \, dx \). (cf. #78, #85, #86)

Sol.) (In general, try to simplify the exponent by using u-substitution.)

Let \( u = -\frac{x^2}{2} \Rightarrow du = -x \, dx \).

\[
\int_0^1 xe^{-x^2} \, dx = \int_0^{-\frac{1}{2}} e^u (-du) = \int_{-\frac{1}{2}}^0 e^u \, du = e^u\bigg|_{-\frac{1}{2}}^0 = 1 - e^{-\frac{1}{2}}
\]

Ex #79. Evaluate \( \int e^{t\sqrt{e^{t+1}}} \, dt \). (cf. #82~84).

Sol.). ( \( \int f(e^t) e^t \, dt = \int f(u) \, du \) for \( u = e^t \). Similar \( u \)-substitutions work for similar problems.)

Let \( u = e^t + 1 \Rightarrow du = e^t \, dt \)

\[
\int e^{t\sqrt{e^{t+1}}} \, dt = \int \sqrt{u} \, du = \frac{2}{3} u^{3/2} + C = \frac{2}{3} (e^t + 1)^{3/2} + C.
\]

Section 7.2. Inverse Functions

Summary. What is \( f^{-1} \)?

Let \( D = \text{domain of } f \).

- \( f \) is called one-to-one if \( a \neq b \Rightarrow f(a) \neq f(b) \).
- The inverse of \( f \) is the unique function \( f^{-1} : R \to D \) satisfying
  \[ f(f^{-1}(y)) = y \quad \forall y \in R, \]
  \[ f^{-1}(f(x)) = x \quad \forall x \in D. \]

\[ D \xrightarrow{f} R \quad \xrightarrow{f^{-1}} \text{ inverts every arrow!} \]

OH: W 2-3 @ MS 2963.
How to find \( f^{-1} \)?

- **(Existence?)** \( f^{-1} \) exists \( \iff \) \( f \) is 1-1.
- **(Formula?)** Set \( y = f(x) \) \( \iff \) \( x = f(y) \) and solve this equation for \( y \).
- **(Graph?)** The graph of \( f^{-1} \) is obtained by reflecting the graph of \( y = f(x) \) around \( y = x \).

\[
\begin{align*}
y &= f(x) \\
y &= x \\
y &= f^{-1}(x)
\end{align*}
\]

(Tip) - Horizontal test.

**(Existence)** If \( f \) is strictly increasing or decreasing \( \Rightarrow \) \( f \) is 1-1.

How to find \( (f^{-1}(x))' \)?

- If \( f \) has the inverse \( g(x) \) and \( f'(g(x)) \neq 0 \), then \( g'(x) \) exists and

\[
g'(x) = \frac{1}{f'(g(x))}.
\]

**Ex. 4.3** Find \( \mathbb{D} \) a domain of \( f(x) = \frac{1}{\sqrt{1 + x^2}} \) on which \( f \) is 1-1.

| a | a formula for \( f^{-1} \). |

**Sol.**

1. We find the regions on which \( f \) is strictly increasing or decreasing.

\[
f'(y) = -\frac{x}{(x^2 + 1)^{3/2}}
\]

- \( \xi (-) \) if \( x > 0 \)
- \( \xi (+) \) if \( x < 0 \)

\[\Rightarrow f \] is strictly decreasing on \( x > 0 \) \n\[\Rightarrow f \] is increasing on \( x < 0 \).

\[\Rightarrow f \] is 1-1 on \( \xi [0, \infty) \) with the range \( (0, 1] \).

2. Let \( y = f(x) = \frac{1}{\sqrt{1 + x^2}} \). Then \( y^2 = \frac{1}{x^2 + 1} \) \( \Rightarrow \frac{1}{y^2} = x^2 + 1 \) \( \Rightarrow x^2 = \frac{1}{y^2} - 1 \)

\[\Rightarrow x = \pm \sqrt{\frac{1}{y^2} - 1}.
\]
Therefore

\[ f^{-1} : \mathbb{R} \to [0,1] \]

\[ g_1(y) = \sqrt{\frac{1}{y} - 1} \]

\[ g_2(y) = -\sqrt{\frac{1}{y} - 1} \]

\[ y = f(x) \implies y = g_1(x), y = g_2(x) \]

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**Ex#19**

Let \( f(x) = x^9 + x + 1 \).

(a) Show that \( f^{-1} \) exists.
(b) What is the domain of \( f^{-1} \)?
(c) Find \( f^{-1}(3) \).

**Sol**

(a) We show that \( f \) is strictly increasing:

\[ f'(x) = 9x^8 + 1 > 0 \quad \text{OK!} \]

(b) Domain of \( f^{-1} = \text{Range of } f \).

\[ \Rightarrow \text{Range of } f = (-\infty, \infty) \]

\[ \Rightarrow \text{Domain of } f^{-1} = (0, \infty) \]

(c) (There is no general method of finding the value of inverse functions. Just try small integer inputs.)

\[ f(1) = 1^9 + 1^3 + 1 = 3 \]

\[ f^{-1}(3) = 1 \]

---

**Ex#24**

Let \( g = \text{inverse of } f(x) = x^3 + 1 \).

(a) Find \( g \)

(b) Calculate \( g'(x) \) both by Thm and by direct calculation.
(a) Let \( y = g(x) \). Then \( x = y^3 + 1 \Rightarrow x - 1 = y^3 \Rightarrow y = \sqrt[3]{x - 1} \).

(b) By Thm:
\[
g'(x) = \frac{1}{f'(g(x))} = \frac{1}{3g(x)^2} = \frac{1}{3} (x - 1)^{-\frac{2}{3}}.
\]

By direct calculation:
\[
g'(x) = \frac{2}{3} (x - 1)^{\frac{1}{3}} (x - 1)^{-\frac{2}{3}} = \frac{1}{3} (x - 1)^{-\frac{3}{2}}. \quad \text{III}
\]

---

**Ex #32**

Find \( g'(-\frac{1}{2}) \) where \( y = \text{inverse of } f(x) = \frac{x^3}{x^2 + 1} \).

Note that \( f(1) = -\frac{1}{2} \Rightarrow g(-\frac{1}{2}) = 1 \). So by Thm,
\[
g'(-\frac{1}{2}) = \frac{1}{f'(g(-\frac{1}{2}))} = \frac{1}{f(1)}.
\]
But since \( f'(1) = \frac{x^4 + 3x^2}{(x^2 + 1)^2} \bigg|_{x = -1} = 1 \), we get \( g'(-\frac{1}{2}) = 1 \). \quad \text{III}

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**Section 7.3** Logarithms and their derivatives

**Summary**

- The logarithm function \( \log_b x \) (\( b > 0, b \neq 1 \)) is the inverse of \( b^x \).

- Graph of \( y = \log_b x \):

  - \( b > 1 \):
    - \( y = \log_b x \)
    - \( 0 < b < 1 \):
    - \( y = \log_b x \)

- The natural logarithm is \( \ln x = \log_e x \).

- (Logarithm Laws)
  1. \( \log_b (xy) = \log_b x + \log_b y \)
  2. \( \log_b \left( \frac{x}{y} \right) = \log_b x - \log_b y \)
  3. \( \log_b (x^n) = n \log_b x \)
  4. \( \log_b 1 = 0, \quad \log_b b = 1 \).
• (Derivatives)
  (i) \((\ln x)' = \frac{1}{x}\)

  (ii) \((\log_b x)' = \frac{1}{x \ln b}\)

  (iii) \((b^x)' = b^x \ln b\).

  (iv) \((\ln f(x))' = \frac{f'(x)}{f(x)}\)

  (v) \(\{\ln(f(x)g(x))\}' = \frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)}\)

  (Logarithmic differentiation)

  (Useful when there are bunch of products)

• (Integrals)
  (i) \(\int \frac{dx}{x} = \ln |x| + C\)

  (ii) \(\int x^2 \frac{dt}{t} = \ln x.\)

Ex #34.

Find \((\ln(t5^t))'\).

\[\ln(t5^t) = \ln t + \ln(5^t) = \ln t + t \ln 5.\]

\[\Rightarrow (\ln(t5^t))' = (\ln t)' + (t \ln 5)' = \frac{1}{t} + \ln 5.\]

Ex #36.

Find \(\frac{d}{dx} e^{(\ln x)^2}\).

By Chain Rule,

\[f(x) = e^x, \quad g(x) = (\ln x)^2,\]

\[\frac{d}{dx} e^{(\ln x)^2} = \frac{d}{dx} f(g(x)) = f'(g(x))g'(x)\]

\[= e^{(\ln x)^2} \cdot \frac{d}{dx} (\ln x)^2\]

\[= e^{(\ln x)^2} \cdot 2 \ln x \cdot \frac{1}{x} = \frac{2 \ln x}{x} e^{(\ln x)^2}.\]

Ex #16.

Find \(\frac{d}{dx} x^{x^2}\).

By Chain Rule.

\[\frac{d}{dx} x^{x^2} = \frac{d}{dx} e^{x^2 \ln x} = e^{x^2 \ln x} \cdot \frac{d}{dx} (x^2 \ln x)\]

\[= x^{x^2} \cdot (2x \ln x + x^2)\]

\[= x(x \ln x + 1) x^{x^2}.\]

Ex #87.

Evaluate \(\int \frac{dx}{2x+4}\) with \(u = x+2\).

1. \(\int \frac{dx}{2x+4} = \frac{1}{2} \int \frac{dx}{x+2} = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln |x+2| + C.\)

2. \(\int \frac{dx}{2x+4} = \int \frac{du}{2u} = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln |2x+4| + C.\)

What happened here? Ans: C are different!
Ex #92 \[ \int \cot x \, dx = ? \]

So \[ \int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \int \frac{du}{u} = \ln |u| + C \]

\[ = \ln |\sin x| + C. \]

Ex #97 \[ \int \frac{(\ln x)^2}{x} \, dx = ? \]

So \[ u = \ln x, \quad du = \frac{dx}{x} \Rightarrow \int \frac{(\ln x)^2}{x} \, dx = \int u^2 \, du = \frac{u^3}{3} + C \]

\[ = \frac{(\ln x)^3}{3} + C. \]