1 Solution

Problem 1.1. Use [MP05] Theorem 2 and the isoperimetric theorem from the class to prove the followings: Let $d \geq 2$ and $p > p_c(\mathbb{Z}^d)$. Given a configuration $\omega$, let $(X_n)$ be a lazy random walk on $\omega$ with the transition matrix

$$p_{\omega}(x,y) = \begin{cases} \frac{1}{2d} & x \sim_\omega y \text{ if } x \neq y \\ 1 - \frac{\deg x}{2d} & \text{if } y = x \end{cases}.$$ 

Then there exists a constant $C = C(p,d) > 0$ and a random variable $\mathcal{N}$ such that for $\mathbb{P}(\cdot \mid 0 \in C_\infty)$-a.s. $\omega$ we have:

(a) $\mathcal{N}(\omega) < \infty$,

(b) $p^n_{\omega}(0,0) \leq Cn^{-d}$ for $n \geq \mathcal{N}(\omega)$.

[Hint. First bound the Cheeger constant of small sets, smaller than $n^{1/3}$, (boundary at least one from connectedness to $\infty$). Second write the integral in Morris Peres as the sum up to $n^{1/3}$ and from $n^{1/3}$ up to the $1/\varepsilon$ factor. And calculate.]

Proof. Before presenting the solution, we review the necessary ingredients:

Lemma 1.1. [BBH 08 Lemma 3.4] Let $d \geq 2$ and $p > p_c(\mathbb{Z}^d)$. Then there exists a constant $c_1 = c_1(p,d) > 0$ and a random variable $R_1 = R_1(\omega)$ with $\mathbb{P}(R_1 < \infty) = 1$ such that for a.s. $\omega \in \{0 \in C_\infty\}$ and $R \geq R_1(\omega)$ the following holds: For any connected $\Lambda \subset C_\infty \cap [-R,R]^d$ with

$$|\Lambda| \geq (\log R)^{d/(d-1)}$$

we have

$$|\partial^\omega \Lambda| \geq c_1 |\Lambda|^{(d-1)/d}.$$

Lemma 1.2. [MP05 Theorem 2] Let $\omega \subset \mathbb{Z}^d$ be an infinite connected subgraph containing 0 and $p_{\omega}(x,y)$ be as in Problem 1.1. Define

$$\phi(r) = \phi(r,\omega) = \inf\{||S||/|S| : S \subset \omega \text{ and } |S| \leq r\}.$$ 

If

$$n \geq 1 + \int_1^{4/\varepsilon} \frac{4du}{u\phi(u)^2},$$

then

$$p^n_{\omega}(x,y) \leq \varepsilon.$$

Let $c > 0$ and $R_1$ be as in Lemma 1.1 and let $\omega \in \{0 \in C_\infty\}$ satisfy the conclusion of Lemma 1.1. For this $\omega$, we want to apply Lemma 1.2. This requires to establish some lower bound of $\phi(r)$. The trick is to modify the graph where outside the range of the random walk. Let $R \geq R_1(\omega)$. Then the range of $(X_n : 0 \leq n \leq R)$ started at 0 is contained in the box $[-R,R]^d$, which means that we can modify our graph $\omega$ outside the box without changing the value of $p^n_{\omega}(0,0)$. Explicitly, let

$$\omega' = (\omega \cap [-R,R]^d) \cup (\mathbb{Z}^d \setminus [-R,R]^d).$$

Then $p^n_{\omega}(0,0) = p^n_{\omega'}(0,0)$. This modification gives us certain advantage that we can bound $\phi(r) = \phi(r,\omega')$ more easily. We work with $\omega'$ in the sequel.
Now let \( S \subset \omega' \) be finite and connected. Then there are 3 different cases:

- **Case 1.** If \( |S| < R^{1/3} \), then we get a trivial bound
  \[
  \frac{|\partial S|}{|S|} \geq \frac{1}{R^{1/3}},
  \]
  where \( \partial S = \partial S' \) is the edge-boundary of \( S \) in \( \omega' \).

- **Case 2.** Assume that \( |S| \geq R^{1/3} \) and
  \[
  |S \cap [-R, R]^d| < \frac{1}{2} |S|.
  \]
  That is, more than half of the volume lies outside the \( R \)-box. Now write \( \mathbb{Z}^d \setminus [-R, R]^d \) as a union of \( 2d \) half-lattices
  \[
  L_{i,+} := \{(x_1, \ldots, x_d) \in \mathbb{Z}^d : \pm x_i > R \}.
  \]
  Then at least one of those \( 2d \) half-lattices contain portion of \( S \) with volume \( \geq \frac{1}{4d} |S| \). Let us pick any such half-lattice and denote it by \( L \). Then by reflecting \( S \cap L \) along the boundary of \( L \), we obtain a figure for which isoperimetric inequality for \( \mathbb{Z}^d \) is applicable.  
  
  \[
  |\partial S| \geq |(\partial S) \cap E(L)| \geq \frac{c_2}{|S \cap L|^1/d} \geq \frac{c_2}{|S|^1/d}.
  \]

- **Case 3.** Assume that \( |S| \geq R^{1/3} \) and
  \[
  |S \cap [-R, R]^d| \geq \frac{1}{2} |S|.
  \]
  Write \( B = [-R, R]^d \) for the \( R \)-box, and recall that \( E(B) \) is the set of edges in \( B \). Then we claim that
  \[
  |\partial S| \geq |\partial (S \cap B)|.
  \]
  Pictorially, the situation is as in Figure 1. Imagine the following situation: for each edge \( e \in (\partial B) \cap \partial (S \cap B) \) we shoot

Figure 1: A typical situation for \((\partial B) \cap \partial (S \cap B)\).

\( ^1 \)Julian proposed this idea to me.
as desired. Now if we write $S \cap B$ as a disjoint union of connected components $S_1', \cdots, S_m'$, then

$$\frac{\partial S}{|S|} \geq \frac{|\partial(S \cap B)|}{2|S \cap B|} \geq \frac{|\partial S_1'| + \cdots + |\partial S_m'|}{4(|S_1'| + \cdots + |S_m'|)} \geq \frac{1}{4} \min_{1 \leq i \leq m} \frac{|\partial S_i'|}{|S_i'|},$$

where the second inequality follows since the boundaries $\partial S_1', \cdots, \partial S_m'$ overlap at most twice. If we apply Lemma 1.1 to components of volume size $\geq R^{1/3}$ and apply (1.1) to the others, we get

$$\frac{|\partial S|}{|S|} \geq \frac{1}{4} \left( \frac{1}{R^{1/3}} \wedge \frac{c_1}{|S|^{1/d}} \right).$$

Combining all the estimations, we find that there exists $c_3 = c_3(d, p) > 0$ such that

$$\frac{|\partial S|}{|S|} \geq \frac{c_3}{R^{1/3} \vee |S|^{1/d}},$$

which implies the following lower bound

$$\phi(r) \geq \frac{c_3}{R^{1/3} \vee r^{1/d}} \quad \text{(1.3)}$$

Using this bound we obtain

$$1 + \int_1^{4/e} \frac{4du}{u \phi(u)^2} \leq 1 + 4c_3^{-2} \int_1^{4/e} (R^{2/3} + u^{2/d}) \frac{du}{u} \leq 1 + 4c_3^{-2} \left( R^{2/3} \log(4/e) + \frac{d}{2} (4/e)^{2/d} \right).$$

Finally, let us plug $R = n$ and $\varepsilon = C/n^{d/2}$ to this estimate, where $C = C(d, p) = 4(2 \sqrt{d}/c_3)^d > 0$ is a constant. If $n \geq R(\omega)$ is sufficiently large, we have

$$1 + \int_1^{4/e} \frac{4du}{u \phi(u)^2} \leq 1 + o(1) \frac{1}{2} n < n$$

and hence $p_n^0(0, 0) \leq \varepsilon = C/n^{d/2}$. This completes the proof. \qed

**Problem 1.2.** Use Problem 1.1 to prove that the random walk on super critical percolation in $\mathbb{Z}^d$, $d \geq 3$, is transient.

**Proof.** Let $d \geq 3$. Then by Problem 1.1 $\mathbb{P}(\cdot \mid 0 \in \mathcal{C}_\omega)$-a.s. $\omega$, we have $\sum_n p_n^0(0, 0) < \infty$. So by the Borel-Cantelli’s Lemma we have

$$\mathbb{P}_\omega(X_n = 0 \text{ infinitely often}) = 0 \quad \mathbb{P}(\cdot \mid 0 \in \mathcal{C}_\omega)$-a.s.$$

This implies the transience as desired. \qed

**Problem 1.3.** Let $(\Omega, \mathcal{F})$ be a measure space and let $Q : \Omega \times \mathcal{F} \to [0, 1]$ be a Markov transition kernel (which means that $Q$ is measurable in the first coordinate and a probability measure in the second coordinate). Given a probability measure $\mu$ on $(\Omega, \mathcal{F})$, let $(X_n : n \geq 0)$ denote a path of the Markov chain with transition kernel $Q$ and initial law $\mathbb{P}(X_0 \in A) = \mu(A)$. Show that, if for any bounded and measurable $f, g : \Omega \to \mathbb{R}$, we have

$$\int \mu(d\omega)Q(\omega, d\eta)f(\omega)g(\eta) = \int \mu(d\omega)Q(\omega, d\eta)g(\omega)f(\eta)$$

then

$$\mathbb{P}(X_0, \cdots, X_n) \overset{\text{law}}{=} (X_n, \cdots, X_0)$$

for each $n \geq 0$. 

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Proof. Define the operator $P$ for bounded measurable functions by

$$P f(\omega) = \int_\Omega f(\eta) Q(\omega, d\eta) = \mathbb{E}_\omega f(X_1),$$

where $\mathbb{E}_\omega$ is the expectation with respect to the law $P_\omega(\cdot) = Q(\omega, \cdot)$. Then the condition (1.4) reads

$$\mathbb{E}(f P g) = \mathbb{E}(g P f).$$

Now let $A_0, \ldots, A_n \in \mathcal{F}$. Then by the Markov property, we get

$$\mathbb{P}_\omega(X_0 \in A_0, \ldots, X_n \in A_n) = 1_{A_0} P(1_{A_1} P(1_{A_2} \cdots P(1_{A_n}))) (\omega).$$

Now by repeatedly applying the condition (1.4),

$$\begin{align*}
\mathbb{P}(X_0 \in A_0, \ldots, X_n \in A_n) &= \mathbb{E}[\mathbb{P}_\omega(X_0 \in A_0, \ldots, X_n \in A_n)] \\
&= \mathbb{E}[1_{A_0} \cdot P(1_{A_1} P(1_{A_2} \cdots P(1_{A_n})))] \\
&= \mathbb{E}[(P1_{A_0}) 1_{A_1} \cdot P(1_{A_2} \cdots P(1_{A_n})))] \\
&= \mathbb{E}[P(1_{A_1}(P1_{A_0})) \cdot 1_{A_2} P(\cdots P(1_{A_n}))] \\
& \vdots \\
&= \mathbb{E}[1_{A_n} P(1_{A_{n-1}} P(1_{A_{n-2}} \cdots P(1_{A_0}))))] \\
&= \mathbb{P}(X_n \in A_0, \ldots, X_0 \in A_n).
\end{align*}$$

This proves the reversibility of $X_n$ as desired. □

References
