1 Solutions

**Exercise 2.1.** Let $\mathcal{D} = \{ f \in C([0, 1]) : f' \in C([0, 1]) \}$ and define $\| f \|_d = \| f \|_\infty + \| f' \|_\infty$. Show that $\mathcal{D}$ is a Banach algebra and that the Gelfand transform is neither isometric nor onto.

**Proof.** It is straightforward to prove that $\mathcal{D}$ is both a normed vector space and an algebra over $\mathbb{C}$. In order to prove that $\mathcal{D}$ is a Banach algebra, we check the followings:

- The norm $\| \cdot \|_d$ on $\mathcal{D}$ is complete. Indeed, let $\{ f_n \}_{n=1}^\infty \subset \mathcal{D}$ be a sequence such that $\sum_n \| f_n \|_d < \infty$. Then both $\sum_n f_n$ and $\sum f'_n$ converges uniformly in $C([0, 1])$ and hence
  \[
  f(x) = \sum_{n=1}^\infty f_n(x) = \sum_{n=1}^\infty \left( f_n(0) + \int_0^x f_n(t) \, dt \right) 
  = \sum_{n=1}^\infty f_n(0) + \int_0^x \sum_{n=1}^\infty f_n(t) \, dt.
  \]
  Here, the interchange of infinite summation and integral is justified by uniform convergence. In particular, this implies that $f' = \sum_n f'_n$. Finally,
  \[
  \left\| \sum_{n=N}^\infty f_n \right\|_d \leq \sum_{n=N}^\infty \left( \| f_n \|_\infty + \| f'_n \|_\infty \right) = \sum_{n=N}^\infty \| f_n \|_d
  \]
  shows that $\sum_n f_n$ converges to $f$ in $\| \cdot \|_d$-norm. This compiles the proof that $\mathcal{D}$ is complete.

- $\| 1 \|_d = \| 1 \|_\infty + \| 0 \|_\infty = 1$. 


• For any \( f, g \in \mathcal{D} \), we have

\[
\|fg\|_d = \|fg\|_\infty + \|f'g + fg'\|_\infty \\
\leq \|f\|_\infty \|g\|_\infty + \|f'\|_\infty \|g\|_\infty + \|f\|_\infty \|g'\|_\infty \\
\leq (\|f\|_\infty + \|f'\|_\infty)(\|g\|_\infty + \|g'\|_\infty) \\
= \|f\|_d \|g\|_d
\]

These altogether complete the proof that \( \mathcal{D} \) is a Banach algebra.

 Following exactly the same argument as in the proof of Proposition 2.3, we can check that the map

\[
\psi : [0, 1] \rightarrow M \text{ given by } \psi(x) = \delta_x \quad \text{where } \delta_x : f \mapsto f(x)
\]

defines a homeomorphism from \([0, 1]\) onto \( M \). Then for any \( f \in \mathcal{D} \),

\[
\|\Gamma f\| = \sup_{\varphi \in M} \|\varphi(f)\| = \sup_{x \in [0, 1]} \|f(x)\| = \|f\|_\infty.
\]

Consequently, for any non-constant \( f \in \mathcal{D} \) we have \( \|\Gamma f\| < \|f\|_d \) and hence \( \Gamma \) cannot be isometric.

Also, since \( C(M) \cong C([0, 1]) \) and \( C([0, 1]) \) contains plenty of non-differentiable functions, \( \Gamma \) cannot be onto. \( \square \)

**Exercise 2.2.** Let \( X \) be a compact Hausdorff space, \( K \) be a closed subset of \( X \), and

\[
\mathcal{I} = \{ f \in C(X) : f(x) = 0 \text{ for } x \in K \}.
\]

Show that \( \mathcal{I} \) is a closed ideal in \( C(X) \). Show further that every closed ideal in \( C(X) \) is of this form. In particular, every closed ideal in \( C(X) \) is the intersection of the maximal ideals which contain it.

**Proof.** Let \( \mathcal{I}_K \) denote the ideal introduced in the problem:

\[
\mathcal{I}_K = \{ f \in C(X) : f(x) = 0 \text{ for } x \in K \}.
\]

Proving that \( \mathcal{I}_K \) is a closed ideal of \( C(X) \) is straightforward. Indeed,

- If \( f, g \in \mathcal{I}_K \), then \( f(x) + g(x) = 0 \) for \( x \in K \) and hence \( f + g \in \mathcal{I}_K \).
- If \( f \in \mathcal{I}_K \) and \( r \in C(X) \), then \( rf(x) = 0 \) for \( x \in K \) and hence \( rf \in \mathcal{I}_K \).
- Assume that \( f \in \text{cl}(\mathcal{I}_K) \). Since the topology on \( C(X) \) is first-countable, there exists a sequence \( \{f_n\}_{n=1}^\infty \subset \mathcal{I}_K \) that converges uniformly to \( f \). Then \( f(x) = \lim_{n \to \infty} f_n(x) = 0 \) for \( x \in K \) and hence \( f \in \mathcal{I}_K \).

Now assume that \( \mathcal{I} \) is any closed ideal in \( C(X) \). In order to prove that \( \mathcal{I} = \mathcal{I}_K \) for some closed subset \( K \), we proceed as follows:
(Step 1: identify the set $K$) Define $K$ by

$$K = \bigcap_{f \in \mathcal{I}} f^{-1}([0]) .$$

Being an intersection of closed sets, $K$ is closed as well. We claim that $\mathcal{I} = \mathcal{I}_K$. It is clear that $\mathcal{I} \subseteq \mathcal{I}_K$, thus it suffices to prove that $\mathcal{I}_K \subseteq \mathcal{I}$.

(Step 2: a useful lemma) Since $X$ is compact and Hausdorff, any closed subset of $X$ is again compact and Hausdorff. Using this property, we can prove the following useful lemma:

**Lemma 1.1.** Let $F$ be any closed subset of $X$ which is disjoint from $K$. Then there exists $\varphi \in \mathcal{I}$ such that

$$0 \leq \varphi(x) \leq 1 \quad \text{for } x \in X \quad \text{and} \quad \varphi(x) = 1 \quad \text{for } x \in F .$$

**Proof of Lemma.** For each $a \in F$, the construction of $K$ allows us to pick a $g \in \mathcal{I}$ such that $g(a) \neq 0$. Since $\bar{g} \in C(X)$, we have $|g|^2 = g\bar{g} \in \mathcal{I}$ and $|g(a)|^2 \neq 0$. Thus if we let

$$h_a = 2|g|^2 / |g(a)|^2 \quad \text{and} \quad U_a = \{ x \in X : h_a(x) > 1 \} ,$$

then $h_a \in \mathcal{I}$ is a non-negative function and $U_a$ is an open neighborhood of $a$.

Notice that the family $\{ U_a \}_{a \in F}$ is an open cover of $F$. But since $F$ is compact, we can choose a finite subcover $U_{a_1}, \ldots, U_{a_n}$. Using this, we define $\varphi_0$ as

$$\varphi_0 := h_{a_1} + \cdots + h_{a_n} \in \mathcal{I} .$$

Then it satisfies $\varphi_0(x) > 1$ for $x \in F$. Moreover, by the pasting lemma, the function

$$r(x) := \min \{ 1, 1/\varphi_0(x) \} = \begin{cases} 1/\varphi_0(x), & \varphi_0(x) \geq 1 \\ 1, & \varphi_0(x) \leq 1 \end{cases}$$

is a well-defined continuous function on $X$. Thus we have

$$\varphi := r\varphi_0 = \min \{ \varphi, 1 \} \in \mathcal{I}$$

and this function satisfies the conclusion of the lemma as desired. ////

(Step 3: proof of $\mathcal{I} = \mathcal{I}_K$) We first prove that any non-negative function in $\mathcal{I}_K$ is also in $\mathcal{I}$. To this end, $f \in \mathcal{I}_K$ be non-negative. For each $\varepsilon > 0$, define

$$F_\varepsilon = \{ x \in X : f(x) \geq \varepsilon \} = f^{-1}((\varepsilon, \infty)) .$$

Then each $F_\varepsilon$ is a closed subset of $X$ disjoint from $K$. Now choose $\varphi_\varepsilon \in \mathcal{I}$ as in Lemma 1.1 with the
choice $F = F_{\varepsilon}$. Then it follows that
\[ \| f - f \varphi_\varepsilon \| = \| (1 - \varphi_\varepsilon) f 1_{\{x \in X \setminus F_{\varepsilon} \}} \| \leq \varepsilon. \]
Since $I$ is closed, this implies that $f \in I$.

Finally, let $f \in I_K$ be arbitrary. Then it is easy to see that all the following functions
\[ g_+ := \max\{\Re f, 0\}, \quad g_- := \max\{-\Re f, 0\}, \]
\[ h_+ := \max\{\Im f, 0\}, \quad h_- := \max\{-\Im f, 0\} \]
are members of $I_K$. Since these functions are non-negative, they are also members of $I$. Therefore
\[ f = g_+ - g_- + i(h_+ - h_-) \in I \] and the proof is complete. □

Exercise 2.4. Let $X$ be a Banach space and $L(X)$ be the collection of bounded linear operators on $X$. Show that $L(X)$ is a Banach algebra.

Proof. ▶ We know from Proposition 1.41 that $L(X)$ is a Banach space. The argument used in the proof of Proposition 1.14 also works here mutatis mutandis, but we present a proof anyhow:

Let $\{A_n\} \subset L(X)$ be a Cauchy sequence. Then for each $f \in X$ the sequence $\{A_n f\}_n$ is Cauchy in $X$ and hence convergent. Let $A$ be defined by the pointwise limit
\[ A f := \lim_{n \to \infty} A_n f. \]
Clearly $A$ is a linear operator on $X$. We claim that $A \in L(X)$ and that $\{A_n\}$ converges to $A$ in $L(X)$. Indeed, the reverse triangle inequality shows that for any $f \in X$,
\[ \|A_m f - A_n f\| - \|A f - A_n f\| \leq \|A_m f - A f\|. \]
Thus taking $m \to \infty$, we have
\[ \lim_{m \to \infty} \|A_m f - A_n f\| = \|A f - A_n f\|. \]
Now for any $\varepsilon > 0$, pick $N$ such that $\|A_m - A_n\| < \varepsilon$ for all $m, n \geq N$. Then
\[ \|A_m f - A_n f\| \leq \varepsilon \|f\| \quad \text{for any } f \in X, \ m, n \geq N. \]
Taking $m \to \infty$ yields
\[ \|A f - A_n f\| \leq \varepsilon \|f\| \quad \text{for any } f \in X, \ n \geq N, \]
which implies that $A - A_n$ is bounded and that $\|A - A_n\| \leq \varepsilon$ for all $n \geq N$. This proves that $A \in L(X)$ and $\{A_n\}$ converges to $A$ in $L(X)$ as desired.

▶ It is also clear that $L(X)$ is equipped with the algebra structure. So it remains to check that the norm
on $\mathcal{L}(\mathcal{X})$ is compatible with this algebra structure on $\mathcal{L}(\mathcal{X})$. But notice that

$$\|1\| = \sup_{\|f\|=1} \|1(f)\| = \sup_{\|f\|=1} \|f\| = 1$$

and that for any $A, B \in \mathcal{L}(\mathcal{X})$,

$$\|AB\| = \sup_{\|f\|=1} \|ABf\| \leq \sup_{\|f\|=1} \|A\|\|B\|\|f\| = \|A\|\|B\|.$$ 

Therefore $\mathcal{L}(\mathcal{X})$ is a Banach algebra. \qed

**Exercise 2.5.** Show that if $\mathcal{X}$ is finite ($>1$) dimensional Banach space, then the only multiplicative linear functional on $\mathcal{L}(\mathcal{X})$ is the zero functional.

**Proof.** Let $n = \dim \mathcal{X} > 1$ and fix a basis $\mathcal{B} = \{f_1, \ldots, f_n\}$ of $\mathcal{X}$. Also for each $1 \leq i, j \leq n$ denote by $E_{ij} \in \mathcal{L}(\mathcal{X})$ the unique linear operator on $\mathcal{X}$ characterized by

$$E_{ij}\left(\sum_k \lambda_k f_k\right) = \lambda_j f_i.$$ 

In other words, $E_{ij}$ is represented by the matrix whose only non-zero entry is the $(i, j)$-entry with value 1. Then $\{E_{ij}\}_{i, j}$ is a basis of $\mathcal{L}(\mathcal{X})$ and thus every $A \in \mathcal{L}(\mathcal{X})$ can be written as

$$A = \sum_{i, j} a_{ij} E_{ij}.$$ 

We also remark that $(a_{ij})$ coincides with the matrix representation of $A$ with respect to the basis $\mathcal{B}$.

- Now assume that there exists a multiplicative linear functional on $\mathcal{L}(\mathcal{X})$, and let $\varphi$ be any such. It is easy to verify that for any $i, j, k, l$ we have $E_{ij} E_{kl} = \delta_{jk} E_{il}$. Consequently,

$$\varphi(E_{ij})\varphi(E_{kl}) = \delta_{jk}\varphi(E_{il}). \quad (1.1)$$

From this relation, we find that

- When $i \neq j$, $(1.1)$ yields $\varphi(E_{ij})^2 = 0$ and hence $\varphi(E_{ij}) = 0$.

- The previous observation forces that there exists $l$ such that $\varphi(E_{il}) \neq 0$ (for otherwise $\varphi$ is the zero functional). Then $(1.1)$ yields $\varphi(E_{il})^2 = \varphi(E_{il})$ and hence $\varphi(E_{il}) = 1$. Then for any $i \neq l$, $(1.1)$ again tells us that $\varphi(E_{il}) = \varphi(E_{il})\varphi(E_{il}) = 0$.

This shows that we must have $\varphi(A) = a_{ll}$ for some $l$. But since $n > 1$, we can choose $A, B \in \mathcal{L}(\mathcal{X})$ such that

$$\varphi(AB) = \sum_k a_{1k} b_{kl} \neq a_{ll} b_{ll} = \varphi(A)\varphi(B).$$
(For example, pick $A$ and $B$ such that $a_{ij} = b_{ij} = 1$ for all $i, j$.) This contradiction implies that no such $\varphi$ exists, hence the statement follows.  $\square$