Final, Math 171, Spring 2015
Instructor: Tonči Antunović

Printed name: ____________________________________________

Signed name: _______________________________________________________________________________________

Student ID number: ____________________________________________

Instructions:

• Read problems very carefully. Please raise your hand if you have questions at any time.

• The correct final answer alone is not sufficient for full credit - try to explain your answers as much as you can. This way you are minimizing the chance of losing points for not explaining details, and maximizing the chance of getting partial credit if you fail to solve the problem completely. However, try to save enough time to seriously attempt to solve all the problems.

• If it’s obvious that you final answers can be simplified, please simplify them. Otherwise, your final answers need to be simplified only if this is required in the statement of the problem.

• You are allowed to use only the items necessary for writing. Notes, books, sheets, calculators, phones are not allowed, please put them away. If you need more paper please raise your hand (you can write on the back pages).

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1. Each of the five parts below is worth 2 points.
   (a) (2 points) If $X_n$ is a simple random walk on $\mathbb{Z}$ started from 0, show that $Y_n = (X_n - 2)X_n^2$ is not a Markov chain.

   finite-state

   (b) (2 points) Consider an irreducible Markov chain started from a state $x$. Let $N$ be the number of times the chain hits another state $y$ before returning back to $x$. Express $E[N]$ in terms of $\pi(x)$ and $\pi(y)$, where $\pi$ is the stationary distribution of the chain.

   (c) (2 points) Write the transition matrix of a chain which has two recurrent and two transient states.

   (d) (2 points) Let $X_n$ be a martingale such that $P(X_0 = 1) = P(X_0 = 0) = 1/2$. Compute the expectation $E[X_n]$ for all $n \geq 1$.

   (e) (2 points) Give an example of a Markov chain on $\mathbb{Z}$ which is not a martingale.
2. Consider the Markov chain on the state space \{1, 2, 3, 4, 5\} with the transition matrix

\[
P = \begin{bmatrix}
0 & 1/2 & 1/2 & 0 & 0 \\
0 & 0 & 1/2 & 1/2 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1/2 & 1/2 \\
0 & 0 & 0 & 2/3 & 1/3 \\
\end{bmatrix}
\]

(a) (5 points) Find all transient and all recurrent states. Find all irreducible components.
(b) (5 points) Compute the limit \(\lim_n p^n(1, 5)\).
3. Consider the Markov chain on the state space \{0, 1, 2, \ldots, n\} such that (n \geq 3)

- \(p(0, i) = 1/n\) and \(p(i, 0) = 1/3\), for all \(1 \leq i \leq n\)
- \(p(i, i - 1) = p(i, i + 1) = 1/3\), for all \(2 \leq i \leq n - 1\);
- \(p(1, n) = p(1, 2) = 1/3\) and \(p(n, n - 1) = p(n, 1) = 1/3\).

(a) (5 points) Compute the limit \(\lim_{k \to \infty} p^k(0, 0)\).

(b) (5 points) Assuming that the chain starts at 1, compute the expected number of steps it takes for the chain to return to the state 1.
4. A figure is placed on the $5 \times 5$ chessboard.

(a) (5 points) Assume that the figure can move only to the squares directly to the left, directly to the right, directly above or directly below (if they exist). Figure decides to perform each of the possible steps with equal probabilities. The game is stopped when the figure jumps onto either the central square or on the squares on the edge of the board (if the figure starts there, no steps are performed). Compute the expected number of steps performed and the probability that the figure ends up in the center. Find the answer for each of the 25 squares on the board.

(b) (5 points) Solve the same type of the problem as in the first part. However, now the figure is only allowed to move diagonally to the closest square, so either to the square directly up-right, square directly up-left, square directly down-left and square directly down-right (as long as the figure doesn't fall off the board). Each of the legal steps are performed with equal probabilities. Also now the figure stops when it jumps onto either one of the squares on the left edge of the board, or one of the squares on the right edge of the board. Find the expected number of jumps performed and the probability that the figure ends up on the right side. Find the answer for each of the 25 squares on the board.
5. Consider a Markov chain on the state space \( \{(n,i) | n \in \mathbb{Z}, n \geq 0, i \in \{0,1\} \} \). Assume that the transition probabilities are given by

- \( p((n,0),(n+1,0)) = \frac{n+1}{n+2}, \quad p((n,0),(n,1)) = \frac{1}{n+2}, \) for all \( n \geq 0 \);
- \( p((n,1),(n-1,1)) = 1, \) for all \( n \geq 1 \);
- \( p((0,1),(0,0)) = 1 \).

(a) (5 points) Show that the chain is irreducible and recurrent.
(b) (5 points) Is the chain positive recurrent? Explain your answer.
6. Consider the continuous Markov chain with the $Q$-matrix

$$Q = \begin{bmatrix} -4 & 2 & 2 \\ 1 & -2 & 1 \\ 3 & 2 & -5 \end{bmatrix}$$

Assume that the chain starts from the state 1.

(a) (5 points) Compute the probability that the chain performs the first jump to 2, then to 3 and then back to 2.

(b) (5 points) Let $T = \min\{t > 0 | X_t = 3\}$ be the first time when the chain is at the state 3. Compute the $E[T]$. 
7. Consider the continuous time Markov chain on the state space \( S = \{1, 2, \ldots, n\} \), whose embedded chain jumps from any state \( i \in S \) to any other state with equal probabilities. Assume that when at the state \( i \), the chain jumps after an exponentially distributed time with the parameter \( 2^{i-1} \).

(a) (5 points) Write the \( Q \) matrix of the chain.

(b) (5 points) Find the stationary distribution of this chain.
8. A box contains one red and one green ball. We recursively perform the process in which we draw one ball from the box (uniformly at random), we return the ball back to the box and we also add another ball of the same color. Let $X_n$ have the value $R$ or $G$ depending on the color of the ball drawn in the $n$-th step.

(a) (5 points) Let $M_n$ denote the proportion of the red balls in the box, after we performed $n$ steps. Show that $M_n$ is a martingale, with respect to $X_n$.

(b) (5 points) Show that the ratio of the red and green balls is not a martingale with respect to $X_n$. 

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