1.2 Multistep Transition Probabilities

**Review**
- \( p^m(i,j) = P(X_{n+m} = j \mid X_n = i) \) does not depend on \( n \) and satisfies

**THM**
- \( [p^m(i,j)] = [p(i,j)]^m \).

(Chapman - Kolmogorov equation)

\[ p^{mn}(i,j) = \sum_k p^m(i,k) p^n(k,j). \]

**Example**

A flea hops from one vertex to another equally likely. Calculate \( p^3(1,1) \).

**Sol.1**

Pictorially, we have

\[ p = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \]

\[ p^2 = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}. \]

\[ p^3(1,1) = \sum_{k=1}^3 p^2(1,k) p(k,1) \]

\[ = \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{4}. \]

**Sol.2**

Lumping 2 and 3 together, we get

\[ \tilde{p} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \Rightarrow \tilde{p}^3 = \begin{pmatrix} \frac{1}{8} & \frac{3}{8} \\ \frac{3}{8} & \frac{1}{8} \end{pmatrix}. \]

\[ p^3(1,1) = \tilde{p}^3(1,1) = \frac{1}{4}. \]
**Example 2**

Toss a fair coin until you first see HH. Find the probability that you don't see HH up to the 3rd toss.

**Solution**

Introduce states:

\[
\begin{align*}
1 &= \text{last toss is not H.} \\
2 &= \text{last toss is H but last 2 tosses is not HH} \\
3 &= \text{we have seen HH.}
\end{align*}
\]

Then pictorially:

![Diagram showing states 1, 2, and 3 connected by arrows]

So the transition matrix \( \mathbf{P} \) is

\[
\mathbf{P} = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} \\
0 & 0 & 1
\end{pmatrix}
\]

This gives

\[
\mathbf{P}^3 = \begin{pmatrix}
\frac{3}{8} & \frac{1}{8} & \frac{3}{8} \\
\frac{2}{8} & \frac{1}{8} & \frac{5}{8} \\
0 & 0 & 1
\end{pmatrix}
\]

Since the initial state is 1, \( \mathbf{P}^3(1,1) + \mathbf{P}^3(1,2) = \frac{5}{8} \).

**Example 3**

Let \( X = (X_n)_{n \geq 0} \) be Markov. Show that \( Y = (Y_n)_{n \geq 0} \), \( Y_n = X_{2n} \) is also Markov. What is the transition probabilities for \( Y \)?

**Solution**

Let \( \mathbf{P} \) be the transition matrix of \( X \). Then

\[
\begin{align*}
P(Y_{n1}=j \mid Y_n=\tilde{c}, Y_{n-1} = \tilde{c}_{n-1}, \ldots, Y_0 = \tilde{c}_0) &= P(X_{n2}=j \mid X_n=\tilde{c}, X_{n-2} = \tilde{c}_{n-2}, \ldots, X_0 = \tilde{c}_0) \\
&= P(Y_{n2}=j \mid Y_{2n}=\tilde{c}) \\
&= P^3(\tilde{c}, j).
\end{align*}
\]

So \( Y \) is Markov w/ transition prob. \( P^3(\tilde{c}, j) \).
**Example 4**

Roll a fair dice repeatedly, and let $Y_n$ the $n$th roll.

1. If $X_n = Y_n + Y_{n-1}$, $n \geq 2$, is $X$ Markov?
2. If $X_n = Y_1 + \cdots + Y_n$, $n \geq 1$, is $X$ Markov?
3. If $X_n = \frac{1}{2} Y_n + \frac{1}{2} Y_{n+1}$, $n \geq 2$, is $X$ Markov?

**Sol.**

1. **No,** since

$$P(X_4 = 8 \mid X_3 = 7, X_2 = 2) = P(Y_4 = 2 \mid Y_3 = 6, Y_2 = 1, Y_1 = 1) = \frac{1}{6}.$$

$$P(X_4 = 8 \mid X_3 = 7, X_2 = 12) = P(Y_4 = 7 \mid Y_3 = 1, Y_2 = 6, Y_1 = 6) = 0.$$

2. **Yes,** since $Y_{n+1}$ is independent of $Y_1, \cdots, Y_n$ and

$$P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \cdots) = P(Y_{n+1} = j - i \mid Y_n = i - i_{n-1}, \cdots, Y_1 = i)$$

$$= P(Y_{n+1} = j - i)$$

$$= P(Y_{n+1} = j \mid X_n = i).$$

3. **Yes,** since whatever the value of $Y_0, \cdots, Y_i$ is, $P(Y_{n+1} = Y_i) = \frac{1}{6}$. 

**Example 5**

Consider a Markov chain $X$ with diagram

![Diagram](image)

Calculate $P^{2015}(0, 2)$.

**Sol.**

There is only one possible history from 0 to 2 in 2015 steps:

$$0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 1 \rightarrow \cdots \rightarrow 1 \rightarrow 2.$$

Therefore $P^{2015}(0, 2) = P(0, 1) P(1, 2) P(2, 3) \cdots = \frac{3}{8}$. 

**Example 6** Consider a Markov chain w/ 2 states 1, 2 such that

\[
\begin{array}{c}
\circ & \overset{2/3}{\xrightarrow{1/4}} & \circ \overset{7/4}{\xleftarrow{1/4}} \circ \\
\end{array}
\]

Assume we know

\[
U = \begin{pmatrix} 3 & -8 \\ 3 & 3 \end{pmatrix}, \quad U \begin{pmatrix} 1/3 & 2/3 \\ 1/4 & 3/4 \end{pmatrix} U^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}.
\]

Use this to calculate the limit

\[
\lim_{n \to \infty} P^{(n)} (i,j).
\]

**Solution**

\[
P^n U^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \end{pmatrix}^{n} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\text{ as } n \to \infty.
\]

Therefore

\[
\lim_{n \to \infty} P^n = U^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} U = \frac{1}{11} \begin{pmatrix} 3 & 8 \\ 3 & 8 \end{pmatrix}.
\]