TRANSFORMS (MOMENT GENERATING FUNCTION)

**DEF** \[ M_X(s) = E e^{sX}. \]

**THM** If \( M_X(s) \) is defined near \( s=0 \), \( E(X^n) = M_X^{(n)}(0) \).

**THM** (Inversion property) The distribution of \( X \) is uniquely determined by \( M_X \) whenever it is defined near \( s=0 \).

**Example** Let \( X \) : RV satisfying \( f_X(x) = \frac{1}{\pi (x^2+1)} \). Show that
\[
M_X(s) = \begin{cases} 
1, & \text{if } s = 0 \\
\frac{1}{\pi}, & \text{if } s \neq 0.
\end{cases}
\]

**Sol:** \( M_X(s) = \int_{-\infty}^{\infty} \frac{e^{sx}}{\pi (x^2+1)} \, dx \). If \( s > 0 \), then \( \lim_{x \to -\infty} \frac{e^{sx}}{\pi (x^2+1)} = 0 \) and \( M_X(s) = \infty \). Similarly, for \( s < 0 \), then \( \lim_{x \to -\infty} \frac{e^{sx}}{\pi (x^2+1)} = \infty \) and \( M_X(s) = \infty \).

**Exercise 1** Let \( X \sim N(0,1) \). Find \( E(X^n) \), \( n \geq 1 \).

**Sol.** We know that \( M_X(s) = e^{s^2/2} = \sum_{n=0}^{\infty} \frac{1}{n!} (s^2)^n = \sum_{n=0}^{\infty} \frac{s^n}{2^n (n/2)!} \).

- On the other hand,
\[ M_X(s) = E e^{sX} = \sum_{n=0}^{\infty} \frac{s^n}{n!} E(X^n). \]
Comparing two identities we get
\[ E(X^n) = \begin{cases} 
n! / 2^n (n/2)! , & \text{if } n \text{ is even} \\
0 , & \text{if } n \text{ is odd}
\end{cases} \]

**Exercise 2** Let \( Y \sim \text{Exp}(2) \) and \( X = e^Y \). Calculate \( E(X) \) and \( \text{Var}(X) \).

**Sol.** \[ E(X) = E e^Y = M_Y(1) = \frac{1}{2-1} = 2. \]

- \( \text{Var}(X) = E(X^2) - (E(X))^2 = E e^{2Y} - 4 = M_X(2) - 4 = \infty. \)
**Exercise 2** Assume $X$ satisfies $M_X(s) = \left(1 + e^s\right) e^{e^s-1}$. Compute $P(X=0)$.

* (1st way) Expanding $M_X(s)$ in terms of $e^s$,

$$M_X(s) = \frac{1}{2e} \left(1 + 2e^s + \frac{3}{2} e^{2s} + \ldots \right).$$

This shows that $X$ is discrete with range $\{0, 1, 2, \ldots, 3\}$ and $P(X=0) = \frac{1}{2e}$.

* (2nd way) Let $Y \sim \text{Bernoulli}(\frac{1}{2})$, $Z \sim \text{Poisson}(1)$. Then we can check:

$$M_X(s) = M_Y(s) M_Z(s).$$

If we further assume $Y \perp Z$, then

$$M_{Y+Z}(s) = M_Y(s) M_Z(s).$$

So by the inversion property,

$$P(X=0) = P(Y+Z=0) = P(Y=0)P(Z=0)$$

$$= \frac{1}{2} \cdot \frac{1}{e} = \frac{1}{2e}.$$

**Exercise 3** If $X \perp Y$ and $M_{X+Y}(s) = e^s M_X(s)$, what is $Y$?

*Sol* $M_Y(s) = e^s$ and thus $Y=1$.

**Exercise 4** If $X$ is discrete and $M_X(s) = e^{2s} M_X(-s)$, show that $P_X(x) = P_X(2-x)$.

*Sol* $M_X(s) = e^{2s} M_X(-s) = e^{2s} \sum_x e^{-sx} P_X(x) = \sum_x e^{s(2-x)} P_X(x)$

$$= \sum_{x'} e^{sx'} P_X(2-x') \quad \text{(where } x' = 2-x) .$$

By the inversion property, $P_X(x) = P_X(2-x)$.

**Exercise 5** Let $Y_0, \ldots, Y_m \sim \text{Bernoulli}(\frac{1}{2})$, and $X = Y_0 + 2Y_1 + \ldots + 2^m Y_m$. What is the distribution of $X$?

*Sol* $M_X(s) = M_{Y_0}(s) M_{Y_1}(2s) \ldots M_{Y_m}(2^ms) = \left(\frac{1+e^s}{2}\right) \ldots \left(\frac{1+e^{2^ms}}{2}\right)$. 
Recall that
\[
(1+x)(1+x^2) \cdots (1+x^{2^n}) = \frac{1 - x^{2^{n+1}}}{1 - x^2} = 1 + x + x^2 + \cdots + x^{2^n}.
\]
This shows that
\[
M_X(s) = \frac{1}{2^n}(1 + e^s + e^{2s} + \cdots + e^{n(2^n)s}).
\]
Therefore
\[
X \sim \text{Discrete Uniform}(0, 1, \ldots, 2^n-1).
\]

**Exercise 6**

\(X \perp Y, \quad X \sim \text{Poisson}(\lambda), \quad Y \sim \text{Poisson}(\mu)\). Show that
\(X + Y \sim \text{Poisson}(\lambda + \mu)\).

**Sol.**
\[
M_{X+Y}(s) = M_X(s)M_Y(s) = e^{\lambda(e^s-1)}e^{\mu(e^s-1)} = e^{(\lambda+\mu)(e^s-1)}.
\]
By the inversion property, \(X + Y \sim \text{Poisson}(\lambda + \mu)\).

**Exercise 7**

Prove that \(f(s) = \frac{1}{1 + s^2}\) cannot be a transform.

**Sol.** If \(f(s) = M_X(s)\) for some RV \(X\), then
\[
f''(0) = M_X^{(2)}(0) = \mathbb{E}(X^2) \geq 0.
\]
But it is easy to check that
\[
f''(0) = -2,
\]
a contradiction!

**Conditional Expectation**

Recall \(\mathbb{E}[X|Y=y]\) is defined insofar:

- If \(Y\) discrete, \(\mathbb{E}[X|Y=y] = \frac{\mathbb{E}[X 1_{Y=y}]}{P(Y=y)}\) if \(P(Y=y) > 0\).

- If \(X, Y\) jointly continuous, \(\mathbb{E}[X|Y=y] = \int_{-\infty}^{\infty} x f_{X,Y}(x,y) \, dx\).

**Def.** \(\mathbb{E}[X|Y]\) is a random variable such that \(\mathbb{E}[X|Y](w) = \mathbb{E}[X|Y=y]\) whenever \(Y(w) = y\).