170B Note

Sangchul Lee

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1 Week 7

1.1 Summary

**Theorem** (classical CLT). Let \((X_i)\) is an i.i.d. sequence of r.v.s with common mean \(\mathbb{E}X_i = \mu\) and variance \(\text{Var}(X_i) = \sigma^2\). Define \(S_n, Z_n\) by

\[
S_n - X_1 + \cdots + X_n, \quad Z_n = \frac{S_n - \mathbb{E}S_n}{\sqrt{\text{Var}(S_n)}} = \frac{S_n - n\mu}{\sigma\sqrt{n}}.
\]

Then \((Z_n)\) converges in law to a standard normal distribution, in the sense that

\[
\lim_{n \to \infty} F_{Z_n}(z) = \lim_{n \to \infty} P(Z_n \leq z) = \Phi(z) \quad \text{for any } z \in \mathbb{R}. \quad (1.1)
\]

Here are some remarks.

- The conclusion of CLT immediately implies that, for any \(Z \sim \mathcal{N}(0,1)\) and for any interval \(I\)

\[
\lim_{n \to \infty} P(Z_n \in I) = P(Z \in I).
\]

The result (1.1) is then a special case corresponding to \(I = (-\infty, z]\).

- In most problems, CLT is used to estimate probability of events for \(S_n = X_1 + \cdots + X_n\). This can be done by rewriting each event in terms of \(Z_n\). For example,

\[
P(S_n \geq a) = P\left(\frac{S_n - \mathbb{E}S_n}{\sqrt{\text{Var}(S_n)}} \geq \frac{a - \mathbb{E}S_n}{\sqrt{\text{Var}(S_n)}}\right) = P\left(Z_n \geq \frac{a - n\mu}{\sigma\sqrt{n}}\right) = P\left(Z \geq \frac{a - n\mu}{\sigma\sqrt{n}}\right),
\]

where \(Z \sim \mathcal{N}(0,1)\) is any standard normal variable. The last quantity can be written either as

\[
1 - \Phi\left(\frac{a - n\mu}{\sigma\sqrt{n}}\right) \quad \text{or} \quad \Phi\left(-\frac{a - n\mu}{\sigma\sqrt{n}}\right).
\]

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An important observation is that the conclusion of CLT (and in general, convergence in law) only concerns the distribution of $Z_n$. In other words, CLT is really a statement about distributions. Random variables are borrowed just to make the statement concise and comprehensible, but they are not an essential part. As a consequence, any sequence of r.v.s $(\tilde{Z}_n)$ satisfying $F_{Z_n} = F_{\tilde{Z}_n}$ leads to the same conclusion.

This is one of the main differences between CLT and LLN, since both WLLN and SLLN are truly statements about random variables.

We will make use of this observation implicitly in one of the homework problems.

1.2 Problems

Exercise 1.1. Two frogs are sitting on the start line of a race. When the race begins, they jump at each second towards the goal. In every jump, they move forward by distance which is uniformly random between 10 inches and 20 inches, independently of each other. Using the Central Limit Theorem, estimate the probability that they are within 50 inches after 150 seconds has passed since the beginning of the race.

Solution. Let $X_1, \cdots, X_{150}$ and $Y_1, \cdots, Y_{150}$ be the length of the jumps. Then the probability in question is

$$P(|(X_1 + \cdots + X_{150}) - (Y_1 + \cdots + Y_{150})| < 50).$$

Now let $\tilde{X}_i = X_i - Y_i$. Then $E\tilde{X}_i = 0$ and $\text{Var}(\tilde{X}_i) = 50/3$. So we have

$$P\left(|\frac{50}{\sqrt{150 \cdot (50/3)}} \cdot \tilde{X}_1 + \cdots + \tilde{X}_{150} < \frac{50}{\sqrt{150 \cdot (50/3)}}\right) = P\left(|\frac{\tilde{X}_1 + \cdots + \tilde{X}_{150}}{\sqrt{150 \cdot (50/3)}} < 1\right) \approx \Phi(1) - \Phi(-1) = 2\Phi(1) - 1.$$

Exercise 1.2. Let $(X_n)$ be a sequence of i.i.d. standard normal variables and $S_n = X_1 + \cdots + X_n$. Provide a numerical value for the limit as $n \to \infty$ for each of the following three expressions.

(a) $P\left(\frac{-n}{10} \leq S_n \leq \frac{n}{10}\right)$.

(b) $P\left(\frac{-\sqrt{n}}{2} \leq S_n \leq \frac{\sqrt{n}}{2}\right)$.

(c) $P\left(-10 \leq S_n \leq 10\right)$.

Solution. (a) The given probability can be written as

$$P\left(\frac{-n}{10} \leq S_n \leq \frac{n}{10}\right) = P\left(-\frac{1}{10} \leq \frac{S_n}{n} \leq \frac{1}{10}\right) = 1 - P\left(\left|\frac{S_n}{n} - 0\right| > \frac{1}{10}\right).$$
By the WLLN, the probability term in the RHS converges to 0. Therefore
\[ \lim_{n \to \infty} P \left( -\frac{n}{10} \leq S_n \leq \frac{n}{10} \right) = 1 - 0 = 1. \]

(b) The given probability can be written as
\[ P \left( -\frac{\sqrt{n}}{2} \leq S_n \leq \frac{\sqrt{n}}{2} \right) = P \left( -\frac{1}{2} \leq \frac{S_n}{\sqrt{n}} \leq \frac{1}{2} \right). \]

By the CLT, we know that this converges to
\[ \lim_{n \to \infty} P \left( -\frac{\sqrt{n}}{2} \leq S_n \leq \frac{\sqrt{n}}{2} \right) = \Phi \left( \frac{1}{2} \right) - \Phi \left( -\frac{1}{2} \right). \]

(In fact, due to our special choice of \((X_n)\), the given probability is exactly equal to the boxed value above. For the sake of generality, however, we did not make use of this fact.)

(c) (1st way) We know that \(S_n \sim \mathcal{N}(0, n)\). Consequently, \(S_n/\sqrt{n}\) is again a standard normal variable and
\[ P(-10 \leq S_n \leq 10) = P \left( -\frac{10}{\sqrt{n}} \leq \frac{S_n}{\sqrt{n}} \leq \frac{10}{\sqrt{n}} \right) = \Phi \left( \frac{10}{\sqrt{n}} \right) - \Phi \left( -\frac{10}{\sqrt{n}} \right). \]

Therefore as \(n \to \infty\)
\[ \lim_{n \to \infty} P(-10 \leq S_n \leq 10) = \Phi(0) - \Phi(0) = 0. \]

(2nd way) The previous solution depends critically on the particular choice of \((X_n)\). Here is a solution that works in general cases but is theoretically more demanding.

Let \(\varepsilon > 0\) be arbitrary. Then for any large \(n\) (or more specifically, for any \(n\) with \((10/\varepsilon)^2 < n\)), we have \(10 < \varepsilon \sqrt{n}\) and hence
\[ P(-10 \leq S_n \leq 10) \leq P(-\varepsilon \sqrt{n} \leq S_n \leq \varepsilon \sqrt{n}) = P \left( -\varepsilon \leq \frac{S_n}{\sqrt{n}} \leq \varepsilon \right). \]

By the CLT, the last limit converges to \(\Phi(\varepsilon) - \Phi(-\varepsilon)\). Thus taking \(n \to \infty\),
\[ \limsup_{n \to \infty} P(-10 \leq S_n \leq 10) \leq \lim_{n \to \infty} P \left( -\varepsilon \leq \frac{S_n}{\sqrt{n}} \leq \varepsilon \right) = \Phi(\varepsilon) - \Phi(-\varepsilon). \]

But since \(\varepsilon > 0\) was arbitrary and the limsup of LHS does not depend on \(\varepsilon\), we can take \(\varepsilon \to 0^+\) and hence
\[ \lim_{n \to \infty} P(-10 \leq S_n \leq 10) = 0. \]

\(\square\)
Exercise 1.3. Let \((X_i)\) be a sequence of independent r.v.s satisfying \(X_i \sim \text{Poisson}(2i - 1)\), and define \(Y_n\) by
\[
Y_n = \frac{(X_1 + \cdots + X_n)}{n} - n.
\]
Show that \((Y_n)\) converges in law to the standard normal distribution.

Solution. Recall that any independent sum of independent Poisson variables is again a Poisson variable.

• If \(\tilde{X}_1, \tilde{X}_2, \ldots\) are independent Poisson r.v.s of rate 1, then the partial sum \(\tilde{S}_n = \tilde{X}_1 + \cdots + \tilde{X}_n\) is a Poisson variable with rate \(n\). Thus if we let \(\tilde{Z}_n\) by
\[
\tilde{Z}_n = \frac{\tilde{S}_n - \mathbb{E}\tilde{S}_n}{\sqrt{\text{Var}(\tilde{S}_n)}},
\]
then by CLT we have
\[
\lim_{n \to \infty} \mathbb{P}(\tilde{Z}_n \leq z) = \Phi(z). \quad \text{for every } z.
\]

• \(S_n = X_1 + \cdots + X_n\) has the Poisson distribution of rate \(1 + 3 + \cdots + (2n - 1) = n^2\). In particular, we have \(\mathbb{E}S_n = n^2\) and \(\text{Var}(S_n) = n^2\) and hence
\[
Y_n = \frac{S_n - \mathbb{E}S_n}{\sqrt{\text{Var}(S_n)}}.
\]
Therefore both \(Y_n\) and \(\tilde{Z}_n\) have the same distribution and hence
\[
\lim_{n \to \infty} \mathbb{P}(Y_n \leq z) = \lim_{n \to \infty} \mathbb{P}(\tilde{Z}_n \leq z) = \Phi(z) \quad \text{for every } z.
\]