Theory

**Def**

- Given \((A_n)_{n=1}^{\infty}\), sequence of events,

\[
\limsup_{n \to \infty} \bigcap_{k=n}^{\infty} A_k = \bigcap_{n=1}^{\infty} A_n = \{ \omega : \omega \in A_n \text{ for infinitely many } n \}.
\]

\[
\liminf_{n \to \infty} \bigcup_{k=n}^{\infty} A_k = \bigcup_{n=1}^{\infty} A_n = \{ \omega : \omega \in A_n \text{ for all but } \text{finitely many } n \}.
\]

- If \(\limsup_{n \to \infty} A_n = \liminf_{n \to \infty} A_n\), we define

\[
\lim_{n \to \infty} A_n = \limsup_{n \to \infty} A_n = \liminf_{n \to \infty} A_n.
\]

**Thm (Fatou)**

\[
P(\liminf_{n \to \infty} A_n) \leq \liminf_{n \to \infty} P(A_n).
\]

**HW 9.1**

Is it always \(\limsup_{n \to \infty} A_n = \emptyset\)?

- \(A_n = \emptyset\ \forall n\), then \(\limsup_{n \to \infty} A_n = \emptyset\) as well.

- If \(A_n\) are disjoint, then \(\forall \omega \in \Omega, \ \omega \in A_n\) for at most one \(n\).

  So \(\limsup_{n \to \infty} A_n = \emptyset\)

  (Here, \(P(E) = \text{length of } E\).)

- Let \(\Omega = [0,1] \) and \(A_n = (0,\frac{1}{n})\). Then \(A_n\) is decreasing and hence

\[
\bigcap_{k=n}^{\infty} A_k = (0, \frac{1}{n}) = A_n \implies \limsup_{n \to \infty} A_n = \bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \emptyset.
\]

This example is worth it notice because \(\sum_{n=1}^{\infty} P(A_n) = \infty\) but \(P(A_n \cap \omega) = 0\).

(I.e., without independence, 1st BC is a mere sufficient condition.)

**HW 9.2**

Find a sequence \(A_n\) for which \(\lim_{n \to \infty} A_n\) does not exist.

**Sol1**

Let \(A_n = \sum_{k=n}^{\infty} (0,\frac{1}{k})\), \(n\) even on \(\Omega = [0,1]\) with \(P = \text{length}\). Then \(\forall \omega \in \Omega\) satisfies \(\omega \in A_n\) for all but finitely many \(n\), but \(\forall \omega \in (0, \frac{1}{2}) \cup (\frac{1}{3}, 1)\) satisfies \(\omega \in A_n\) for infinitely many \(n\). So

\[
\limsup_{n \to \infty} A_n = (0, \frac{1}{2}) \cup (\frac{1}{3}, 1) \quad \text{but} \quad \liminf_{n \to \infty} A_n = \emptyset.
\]

Consequently, \(\lim_{n \to \infty} A_n\) does not exist.
We claim that

\[ \liminf A_n = \bigcap_{n=1}^{\infty} A_n, \quad \text{and} \]
\[ \limsup A_n = \bigcap_{n=1}^{\infty} A_n. \]

Indeed, notice that

\[ \bigcap_{k=n}^{\infty} A_k = \bigcap_{k=n} A_k. \]

Indeed,

\[ (A_1 \cap \cdots \cap A_n) \cap \bigcap_{k=n}^{\infty} A_k = A_n \cap \bigcap_{k=n}^{\infty} A_k = \bigcap_{k=n} A_k. \]

So we have

\[ \liminf A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n} A_k = \bigcup_{n=1}^{\infty} A_k = \bigcap_{k=n} A_k. \]

\[ \bigcup_{k=n} A_k = A_n, \text{ since } A_k \subset A_n \quad \forall k \geq n. \]

So we have

\[ \limsup A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k = \bigcap_{n=1} A_n. \]

\[ (A_n) \text{ decreasing } \Rightarrow \ P(\lim A_n) = \lim P(A_n). \]

\[ (A_n) \text{ decreasing } \Rightarrow \ P(\lim A_n) = \lim P(A_n). \]

\[ \liminf P(A_n) \geq P(\liminf A_n) = P(\lim A_n) \quad \text{by Fatou's lemma} \]

\[ \limsup P(A_n) \leq P(\limsup A_n) = P(\lim A_n) \quad \text{by Inverse Fatou's lemma}. \]

\[ \limsup P(A_n) = \liminf P(A_n) = P(\lim A_n) \quad \text{shows that } \lim P(A_n) \]

exists and is equal to \( P(\lim A_n) \).

Remark: This proof continues to work under the assumption that \( \lim A_n \) exists.  

Why 'independence' condition required for 2nd BC?

Consider \( \Omega = [0,1] \), \( P= \text{length} \) and \( A_n = (a_n, 1) \). Then \( \limsup A_n = \phi \)

and thus \( P(\limsup A_n) = 0 \). But \( \sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty \).
**Remark** Independence condition guarantees that events occur in a very evenly manner. So if some outcomes happen to occur infinitely many, then it is very likely that all the possible outcome also occurs infinitely many.

On the other hand, in our example, the situation is quite extreme since once an outcome \(\omega \in \Omega\) happens not to occur at \(A_n\), then it does not occur for all the following events \(A_n, A_{n+1}, A_{n+2}, \ldots\).

---

**Continuous RVs**

**Supp 3.18**

* \(P \sim \text{Uniform} [0, 1]\)

* On a given day, a particular machine is functional \(\text{w/ prob} = P\).

* Conditioned on \(P = p\), status of the machine on different days is \(\text{indep.}\)

Then

(a) Find \(P(\text{machine is functional on a particular day})\)

(b) Knowing that the machine was functional on \(m\) out of \(n\) days, find the conditional PDF of \(P\).

---

**Sol**

(a) Let us call this event \(A\). Then

\[
P(A) = \int_0^1 P(A \mid P = x) f_P(x) \, dx
= \int_0^1 x f_P(x) \, dx = EP = \frac{1}{2}.
\]

(b) Let us call this event \(B\). Then

\[
P(B \mid P = x) = \binom{n}{m} x^m (1-x)^{n-m}.
\]

\[
\Rightarrow P(B \mid P \leq p) = \int_0^p P(B \mid P = x) f_P(x) \, dx = \int_0^p \binom{n}{m} x^m (1-x)^{n-m} \, dx.
\]

\[
\Rightarrow P(P \leq p \mid B) = \frac{P(B \mid P \leq p) P(P \leq p)}{P(B)} = \frac{P(B \mid P \leq p) P(P \leq p)}{P(B)} = \frac{\left( \int_0^p \binom{n}{m} x^m (1-x)^{n-m} \, dx \right) \cdot p}{\int_0^1 \binom{n}{m} x^m (1-x)^{n-m} \, dx}.
\]
$$= \binom{n+1}{m} \left( \begin{array}{c} n \in \mathbb{N} \\ m \in \mathbb{N} \end{array} \right) \int_0^p x^m (1-x)^{n-m} \, dx.$$ \(\Rightarrow f_{\text{PLB}}(p) = \frac{\frac{d}{dp} \mathbb{P}(P \leq p | B)}{\epsilon} \)

$$= \frac{(n+1)!}{m! (n-m)!} \frac{d}{dp} \left( p \int_0^p x^m (1-x)^{n-m} \, dx \right).$$

**Example**

Let \( X, Y : \text{i.i.d.} \sim \text{Exp}(\lambda), Z = X + Y, \lambda > 0 \). Find the conditional PDF \( f_{x|z}(x|z) \).

**Sol.**

- \( f_{x|z}(x|z) = \frac{f_{z|x}(z|x) f_X(x)}{f_Z(z)} = \frac{f_Y(z-x) f_X(x)}{f_Z(z)} = \frac{\lambda e^{-\lambda(z-x)} \cdot \lambda e^{\lambda x}}{f_Z(z)} \)
  
  $$= \frac{\lambda^2 e^{-\lambda z}}{f_Z(z)} \quad \text{if} \ 0 < z < x,$$

- \( f_{x|z}(x|z) = 0 \quad \text{if} \ x \notin (0, z).$$

Since \( \frac{\lambda^2 e^{-\lambda x}}{f_Z(x)} \) does NOT depend on \( x \) on \((0, z)\), it is constant and hence by the normalization condition we get

$$\frac{\lambda^2 e^{\lambda x}}{f_Z(x)} = \frac{1}{x}.$$

Consequently

$$f_{x|z}(x|z) = \begin{cases} \frac{1}{z} & \text{if} \ 0 < x < z \\ 0, & \text{otherwise}. \end{cases}$$

(We also obtain \( f_Z(x) = \lambda^2 x e^{-\lambda x} \) as a by-product.)

**Supp 3.19**

If \( X : \text{continuous RV} \) with PDF \( f_X \), find the PDF of \( |X| \).

**Sol.**

- \( f_{|X|}(x) = \frac{d}{dx} \mathbb{P}(|X| \leq x) = \frac{d}{dx} \mathbb{P}(-x \leq X \leq x) \)
  
  $$= \frac{1}{dx} \int_{-x}^x f_X(t) \, dt = f_X(x) + f_X(x), \text{ for } x \geq 0.$$

- \( f_{|X|}(x) = 0 \quad \text{for} \ x < 0. $$

$$\text{III}$$
Example \( X, Y \) indep., \( \sim N(0,1) \). Find the range and PDF of

\[
Z = \frac{(X+Y)^2}{X^2+Y^2}.
\]

Sol.) Write \((X, Y) = (R \cos \Theta, R \sin \Theta)\) using polar coordinates. Then

\[
Z = 1 + \frac{2XY}{X^2+Y^2} = 1 + 2 \sin \Theta \cos \Theta
\]

\[
= 1 + \sin 2\Theta, \quad \in [0,2]\, \text{ : range}.
\]

But we find that, if we restrict the range of \( \Theta \) onto \([0,2\pi]\), then

\[
P(\Theta \leq t) = \int_{r=0}^{t} \int_{0}^{\infty} f_{X,Y}(x,y) \, dx \, dy
\]

\[
= \int_{0}^{t} \int_{0}^{\infty} f_{X,Y}(r \cos \Theta, r \sin \Theta) \, r \, dr \, d\Theta
\]

\[
= \int_{0}^{t} \frac{d\Theta}{2\pi} = \frac{t}{2\pi}, \quad \text{for } 0 \leq t \leq 2\pi
\]

and hence \( \Theta \sim \text{Uniform } [0,2\pi] \). So we have

\[
f_Z(x) = \frac{d}{dx} P(1 + \sin 2\Theta \leq x).
\]

To calculate this we solve

\[
1 + \sin 2\Theta = x \quad \iff \quad \sin 2\Theta = x - 1
\]

\[
\iff 2\Theta = \arcsin(x-1) + (-1)^n \pi.
\]

So in this case,

\[
P(1 + \sin 2\Theta \leq x) = \frac{1}{2} + \frac{1}{\pi} \arcsin(x-1)
\]

\[\Rightarrow f_Z(x) = \frac{1}{\pi \sqrt{x(2-x)}}, \quad 0 \leq x \leq 2.\]